Topics in topology

Lectures by Michael Hutchings, notes by Søren Fuglede Jørgensen

Math 276, UC Berkeley, Fall 2010

Contents

Disclaimer

These are notes from a course given by Michael Hutchings in 2010.[1] They have been written and TeX'ed during the lectures and most parts have not been proofread, so there are bound to be a number of typos and mistakes that should be attributed to me rather than the lecturer. Also, I've made these notes primarily to be able to look back on what actually happened myself, and to get experience with TeX'ing live. That being said, feel very free to send any comments and/or corrections to `fuglede@imf.au.dk`.

1st lecture, August 31st 2010

1 Introduction

During the 1970s, physicists and mathematicians considered various objects from different points of view. What physicists called gauge fields, mathematicians considered as connections of principal bundles. Similarly, both commuties were interested in 4-manifolds; physicists looked at gauge fields on 4-manifolds and mathematicians tried to classify 4-manifolds. Of interest was the anti-self-dual equation; for a $SU(2)$-principal bundle over a 4-manifold X, we look for connections A in this principal bundle satisfying $*F_A = -F_A$, where F_A is the curvature of the connection, and $*$ denotes the Hodge star. Such connections minimize the integral $\int_X |F_A|^2$.

In the following, let X be a simply-connected (closed, oriented, connected) topological 4-manifold. Consider the *intersection form* $Q : H_2(X) \otimes H_2(X) \to \mathbb{Z}$ given by

$$Q(\alpha, \beta) = \int_X \mathrm{PD}(\alpha) \smile \mathrm{PD}(\beta),$$

where $\mathrm{PD} : H_2(X) \to H^2(X)$ denotes the Poincaré dual operator. It is a fact that Q determines the homotopy type of X.

Theorem 1 (Freedman, 1982). *Q also determines the homeomorphism type of X (up to the Kirby–Siebenmann invariant in $\mathbb{Z}_2$).*

Remark 2. In fact, if the Kirby–Siebenmann invariant of a topological manifold is 0, there is no smooth structure on the manifold.

In the 1980s, Donaldson used the moduli space of anti-self-dual connections to define invariants of smooth 4-manifolds and proved the following.

Theorem 3. *If X is a simply-connected smooth 4-manifold, and Q is negative-definite, then Q is diagonalizable.*

We thus have many 4-manifolds without any smooth structures. Also, the same topological 4-manifold may have non-diffeomorphic smooth structures; for example, there are uncountably many non-diffeomorphic smooth structures on $\mathbb{R}^4$.

In his 1994 paper, [Wit], Witten introduced new 4-manifold invariants known now as Seibert–Witten invariants, that count the number of solutions to the Seiberg–Witten equalities. These new invariants are easier to handle than the previously used ones and could be used to re-prove the old theorems as well as produce new ones. One example is the following.

Theorem 4 (The Thom conjecture, Kronheimer–Mrowka, 1994). *Let Σ be a closed, orientable, and connected surface in $\mathbb{C}P^2$ representing $d \in H_2(\mathbb{C}P^2) = \mathbb{Z}$. Then the genus of Σ satisfies*

$$g(\Sigma) \geq \frac{(d-1)(d-2)}{2}.$$

[1] The course homepage is located at `http://math.berkeley.edu/~hutching/teach/276/index.html` – that probably won't be true forever though.

Corollary 5 (The Milnor conjecture)**.** *The unknotting number of the torus not $T_{p,q}$ is $(p-1)(q-1)/2$.*

Remark 6. A combinatorial proof of this corollary was given by Rasmussen using Khovanov homology.

Theorem 7 (The generalized Thom conjecture, Oszváth–Szabó, 2000)**.** *In a closed symplectic 4-manifold, a symplectic surface is genus-minimizing in its homology class.*

In 1989, Floer introduced what is known as instanton Floer homology of 3-manifolds. On a 4-manifold Y, it counts the anti-self-dual connections on $\mathbb{R} \times Y$, which corresponds to gradient flowlines of the Chern–Simons functional. Several versions of Floer homology exist. The one we will consider in this course is the so-called Seiberg–Witten Floer homology (SWF). SWF is constructed rigorously in the book by Kronheimer and Mrowka, [KM]; we will see how the Seiberg–Witten invariants of a 4-manifold X can be calculated by cutting along 3-manifolds and calculating a certain pairing in SWF.

Other versions include the 2001 construction of Heegaard–Floer homology of 3-manifolds by Ozsváth and Szábo, which is conjectured to be isomorphic to Seiberg–Witten theory (which is proved in 3 dimensions), and embedded contact homology (ECH) of 3-manifolds defined using contact geometry. Taubes proved in 2008 that ECH is isomorphic to SWF. This implies the Weinstein conjecture in three dimensions; see the survey article [Hut] by Hutchings.

Using instanton Floer homology, Kronheimer and Mrowka proved the following.

Theorem 8 (Kronheimer–Mrowka, 2010)**.** *Khovanov homology detects the unknot.*

Remark 9. It is still unknown whether or not the Jones polynomial is an unknot-detector. Note that Khovanov homology contains the Jones polynomial.

The plan for the present course is to first go through relevant background material. This includes Morse homology, connections on principal bundles, spin-c structures and Dirac operators (see [LM]). We will then introduce the Seiberg–Witten invariant of 4-manifolds and the Seiberg–Witten Floer homology of 3-manifolds. If time permits, we will consider applications of the theory, including the Weinstein conjecture.

2nd lecture, September 2nd 2010

2 Morse homology

2.1 Definition of Morse homology

Let X be a closed smooth manifold of dimension n. Let f be a smooth function $f : X \to \mathbb{R}$. A critical point of f is a point $p \in X$ such that $df_p = 0$ (that is, a point where all partial derivatives vanish). We want to relate the topology of X to the critical points of f; for example: Given a manifold, what is the minimal number of critical points, a smooth function can have? For example, on the sphere, you need at least two critical points (a given function must have a max and a min). On a torus, the height function has four critical points; this can be cut down to three, though.

Let p be a critical point. Define the *Hessian* $H(f,p) : T_pX \to T_p^*X$ as follows: The differential of f is a map $df : X \to T^*X$, and $H(f,p)$ is defined to be the composition $T_pX \overset{d(df)}{\to} T_{(p,0)}(T^*X) = T_pX \oplus T_p^*X \overset{\text{proj}}{\to} T_p^*X$. An alternative description is $H(f,p) : T_pX \otimes T_pX \to \mathbb{R}$ defined as follows: Let $x^1, \dots, x^n$ be local cooordinates around p. Then $H(f,p) = \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_{i,j}$ (exercise: Check that this is the same as the above). In particular, this is symmetric. We say that p is *non-degenerate* if this matrix does not have 0 as an eigenvalue. Equivalently, the map $df : X \to T^*X$ is transverse to the zero section at $(p,0)$ (see Fig. 1). If p is non-degenerate, let the index $\mathrm{ind}(p)$ of p be the number of negative eigenvalues. For example, with $\mathrm{ind} = 0$, $f \sim x_1^2 + x_2^2$, with $\mathrm{ind} = 2$, $f \sim -x_1^2 - x_2^2$, and with $\mathrm{ind} = 1$, $f \sim x_1^2 - x_2^2$.

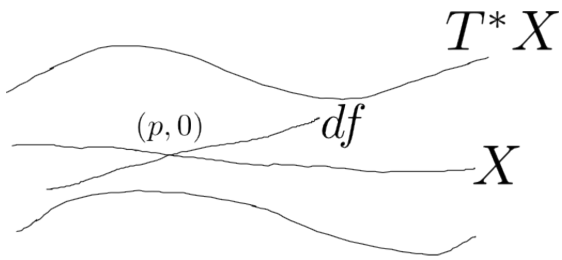

Figure 1: The section df in T^*X is transverse to the zero section.

Definition 10. We call f a *Morse* function, if all its critical points are nondegenerate.

Note that critical points are isolated – in particular, there are only finitely many of them on a compact manifold. A generic smooth function is Morse: That is, the set of Morse functions is an open dense set in the space of all smooth functions $X \to \mathbb{R}$.

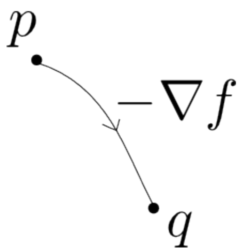

Figure 2: The gradient flow from p to q.

Let f be a Morse function. Let c_i be the number of index i critical points. Let b_i denote the rank of $H_i(X)$. Then $c_i \geq b_i$: Given a metric g on X, define the chain groups of a chain complex $C_*^{\text{Morse}}(f, g)$ by $C_i = \mathbb{Z}\{\text{critical points of index } i\}$. The construction of the differential requires some work. Choose a (generic) metric g on X. If p and q are two critical points, a *(downward) gradient flow line* from p to q is a map $\gamma : \mathbb{R} \to X$ such that $\frac{d\gamma(t)}{dt} = -\nabla f$ and $\lim_{t\to-\infty} \gamma(t) = p$, $\lim_{t\to\infty} \gamma(t) = q$ (see Fig. 2 – here γ should be thought of as following the path of steepest descent from p to q). Let $m(p, q)$ be the set of gradient flowlines from p to q. We want to relate the homology of X to the space of gradient flowlines between critical points. Note that we have an $\mathbb{R}$-action on $m(p, q)$ by shifting the parameter t. Let $\check{m}(p, q) = m(p, q)/\mathbb{R}$ be the set of unparametrized flow lines obtained by modding out by this action.

Claim 11. *If g is generic and $p \neq q$, then $\check{m}(p, q)$ is a manifold of dimension ind(p) − ind$(q) - 1$.*

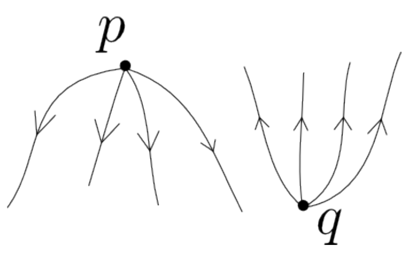

Figure 3: The spaces U_p and S_q.

Define $\psi_t : X \to X$ to be the time t flow of $-\nabla f$. Define the *unstable manifold* $U_p = \{x \in X \mid \lim_{t\to-\infty} \psi_t(x) = p\}$. Define the *stable manifold* $S_q = \{x \in X \mid \lim_{t\to\infty} \psi_t(x) = q\}$ (see Fig. 3). U_p is then an embedded open ball of dimension equal to ind(p). The tangent space

T_pU_p is the negative eigenspace of $H(f,p) : T_pX \to T_p^*X \xrightarrow{g} T_pX$. This is easy to see locally; e.g. $f = -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2$ and $g = dx_1^2 + \cdots dx_n^2$. In general this fact requires some analysis. Similarly, S_q is an embedded open ball of dimension $n - \operatorname{ind}(q)$ and its tangent space will be the positive eigenspace. Observe that $m(p,q) = U_p \cap S_q$ by the identification $\gamma \mapsto \gamma(0)$. It is a fact that if g is generic, then U_p is transverse to S_q; later, we will talk about how to proof statements like these. We call (f,g) *Morse-Smail* if f is Morse and U_p is transverse to S_q for all critical points p and q. For now, assume this condition. Now, we can prove the above claim, as

$$\begin{aligned} \dim m(p,q) &= \dim(U_p \cap S_q) = \dim(U_p) + \dim(S_q) - n \\ &= \operatorname{ind}(p) + (n - \operatorname{ind}(q)) - n = \operatorname{ind}(p) - \operatorname{ind}(q) \end{aligned}$$

The claim about $\dim \check{m}(p,q)$ follows. We can now define a differential $\partial : C_i \to C_{i-1}$ on our chain complex as follows: If p is a critical point of index i, we define

$$\partial p := \sum_{\operatorname{ind}(q)=i-1} \#\check{m}(p,q) \cdot q,$$

where the count is signed. There are three immediate issues with this definition: 1) Why is $\check{m}(p,q)$ finite? 2) What are the signs? 3) Why is $\partial^2 = 0$?

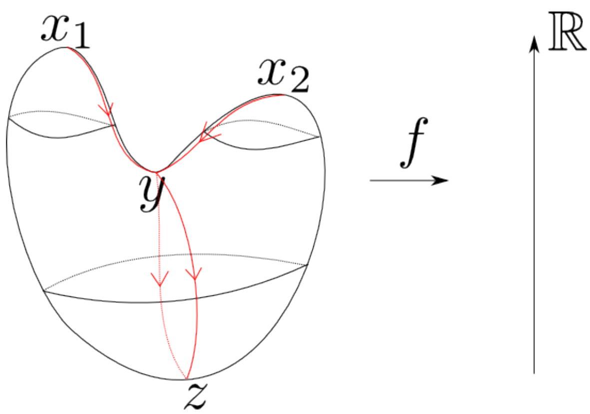

Figure 4: The height function f in the example.

Before dealing with these issues, we consider an example. Let $f : S^2 \to \mathbb{R}$ be the height function in figure 4. Then $C_2 = \mathbb{Z}\{x_1, x_2\}$, $C_1 = \mathbb{Z}\{y\}$, $C_0 = \mathbb{Z}\{z\}$. We have $\partial x_1 = \pm y$, and $\partial x_2 = \pm y$. The two flowlines from y to z have opposite signs as we will see later, so that $\partial y = 0$, $\partial z = 0$. It follows that $H_0 = \mathbb{Z}\{z\}$, $H_1 = 0$ and $H_2 = \mathbb{Z}\{\pm x_1 \pm x_2\}$.

Theorem 12 (Fundamental theorem of Morse theory (according to Hutchings but, in his words, not necessarily to others))**.** *There is a canonical isomorphism $H_*^{Morse}(f,g) = H_*(X)$. Furthermore, this is true for any coefficients.*

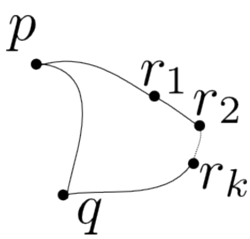

Figure 5: Compactifying $\check{m}(p,q)$.

Concering 1), we consider compactness: For any two critical points $p \neq q$, $\check{m}(p,q)$ need not be compact, but it does have a compactification $\overline{m}(p,q)$, and

$$\overline{m}(p,q) \setminus \check{m}(p,q) = \bigcup_{k \geq 1} \bigcup_{r_1,\dots,r_k} \check{m}(p,r_1) \times \check{m}(r_1,r_2) \cdots \times \check{m}(r_k,q),$$

where the last union is over distict critical points $r_1,\dots,r_k$ not equal to p or q (see Fig. 5 – in compactifying we need to add in sort of broken flowlines). In the above example, $\check{m}(x_1,z) \cong (0,1)$ and $\overline{m}(x_1,z) = \check{m}(x_1,z) \cup \check{m}(x_1,y) \times \check{m}(y,z) \cong [0,1]$, as we add two broken flowlines. In fact, this is not too hard to prove. If $\operatorname{ind}(p) - \operatorname{ind}(q) = 1$, then $\overline{m}(p,q) = \check{m}(p,q)$; if we did have a broken flowline from p to q, this would go to a point with index at least as big, which is impossible by transversality. So in this case, $\check{m}(p,q)$ is compact and therefore finite. If $\operatorname{ind}(p) - \operatorname{ind}(q) = 2$,

$$\overline{m}(p,q) - \check{m}(p,q) = \bigcup_{\operatorname{ind}(r)=\operatorname{ind}(p)-1} \check{m}(p,r) \times \check{m}(r,q).$$

It is a fact that $\overline{m}(p,q)$ is a compact 1-manifold with boundary, and $\partial\overline{m}(p,q) = \bigcup_r \check{m}(p,r) \times \check{m}(r,q)$. In this case, we can proof that $\partial^2 = 0$ (with $\mathbb{Z}_2$ coefficients, ignoring signs). Let $\operatorname{ind}(q) = \operatorname{ind}(p) - 2$. We have

$$\begin{aligned}
\langle \partial^2 p, q \rangle :&= \sum_{\operatorname{ind}(r)=\operatorname{ind}(p)-1} \langle \partial p, r \rangle \langle \partial r, q \rangle \\
&= \sum_{\operatorname{ind}(r)=\operatorname{ind}(p)-1} \#\check{m}(p,r) \cdot \#\check{m}(r,q) \\
&= \# \bigcup_r \check{m}(p,r) \times \check{m}(r,q) \\
&= \#\partial\overline{m}(p,q) = 0,
\end{aligned}$$

with the last equality coming from the fundamental theorem of differential geometry. This proves that $\partial^2 p = 0$.

Concerning 2), the signs, choose an orientation of U_p for each critical point p. Let $\gamma \in m(p,q) = U_p \cap S_q$. There is an exact sequence

$$0 \to T_\gamma m(p,q) \to T_\gamma U_p \to N_\gamma S_q \to 0,$$

and $N_\gamma S_q \cong T_q U_q$ from the orientation on U_p. These orientations induce an orientation on $m(p,q)$. We have another exact sequence

$$0 \to \mathbb{R} \to T_\gamma m(p,q) \to T_\gamma \check{m}(p,q) \to 0,$$

assuming $\mathbb{R}$ is generated by $-\nabla f$, which induces an orientation on $\mathbb{R}$; therefore, we also get an orientation on $\check{m}(p,q)$. The chain complex $C_*(f,g)$ depends on the choice of orientation of U_p, but the homology does not.

3rd lecture, September 7th 2010

Last time we did the following: Suppose X is a closed n-dimensional smooth manifold, $f : X \to \mathbb{R}$ a Morse function and g a generic metric on X. Then we can define a Morse complex $C_*(X,f,g)$, where $C_i = \mathbb{Z}\{\text{critical points of index } i\}$ and $\partial : C_i \to C_{i-1}$ is given by

$$\partial p = \sum_{q \text{ index } i-1} \#\check{m}(p,q) \cdot q,$$

where $\check{m}(p,q)$ is the set of unparametrized downward-flowing flowlines from p to q. Recall that $m(p,q) = U_p \cap S_q$. For every critical point p choose orientation of U_p to obtain a short exact sequence

$$0 \to T_\gamma m(p,q) \to T_p U_p \to N_q S_q \to 0.$$

As $N_q S_q \cong T_q U_q$ an orientation of U gives an orientation of $m(p,q)$. We then get an orientation of $\check{m}(p,q)$ using the short exact sequence

$$0 \to \mathbb{R} \to m(p,q) \to \check{m}(p,q) \to 0.$$

2.2 Properties of Morse homology

Theorem 13. *The homology of the chain complex is canonically isomorphic to the singular homology of X.*

Roughly speaking, the proof is that we get a CW-complex on X using the compactification of the unstable manifold, and this one corresponds to a cellular complex on X. We could imagine some other planet, where they prefer calculus to combinatorics and discovered the Morse complex before any other homology [Hutchings going on about Martian Fields medals ...]. They would have to understand why this will be an invariant of the manifold. So, can we give an a priori proof that $H_*^{\mathrm{Morse}}(X,f,g)$ depends only on X? This will be a warm-up for showing that Floer homology is going to be a topological invariant.

Let $(f_0,g_0),(f_q,g_1)$ be two Morse–Smale pairs. Choose a (generic (which we will see what means later)) homotopy between them $\{(f_t,g_t) \mid t \in [0,1]\}$, where the f_t are smooth functions on X and g_t metrics; we don't require that the (f_t,g_t) are Morse–Smail pairs. Fix a function $h : [0,1] \to \mathbb{R}$ with $h(0) = h(1) = 0$, $h(t) > 0$ for $t \in (0,1)$ and $h'(0) > 0$, $h'(1) < 0$. Define $\phi : C_*^{\mathrm{Morse}}(X,f_0,g_0) \to C_*^{\mathrm{Morse}}(X,f_1,g_1)$ as follows: If $p_0 \in \mathrm{Crit}(f_0)$ and $q_1 \in \mathrm{Crit}(f_1)$, define

$$m(p_0,q_1) = \{\gamma = (t,x) : \mathbb{R}_s \to [0,1] \times X \mid \frac{dt}{ds} = h(t), \frac{dx}{ds} = -\nabla^{g_t} f_t\}$$

(not to be confused with the $m(p,q)$ from before). In other words, these are flow lines of $h\frac{\partial}{\partial t} - \nabla^{g_t} f_t$

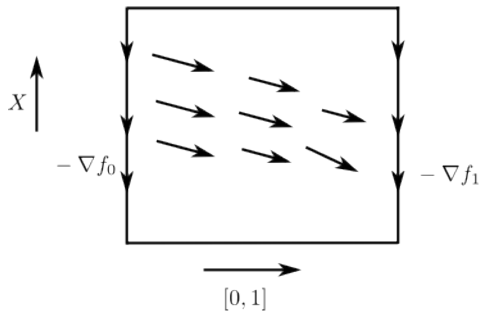

Figure 6: Flowlines on $X \times [0,1]$.

on $[0,1] \times X$ (see Fig. 6). Now, assume p_0 has index i and set

$$\phi p_0 = \sum_{q_1 \text{ index } i} \#\check{m}(p_0,q_1) \cdot q_1$$

with the sign convention from before.

Claim 14. *The map ϕ is a chain map. That is, $\partial_1 \circ \phi = \phi \circ \partial_0$.*

Proof. Analogous to $\partial^2 = 0$; we consider flowlines from index i points to index $i-1$ points and how broken flowlines behave. $\square$

We get an induced map $\Phi : H_*^{\text{Morse}}(X, f_0, g_0) \to H_*^{\text{Morse}}(X, f_1, g_1)$, called a continuation map.

Claim 15. *The map Φ does not depend on the homotopy from (f_0, g_0) to (f_1, g_1).*

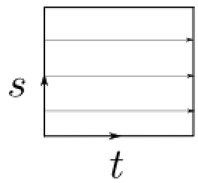

Figure 7: A vector field for a fixed point in X.

Proof. The following argument is due to Floer: Let $\{(f_{0,t}, g_{0,t}\}$ and $\{(f_{1,t}, g_{1,t}\}$ be two homotopies. Choose a (generic) homotopy of homotopies $\{(f_{s,t}, g_{s,t}) \mid s, t \in [0,1]\}$. Look that the vector field $h(t)\frac{\partial}{\partial t} - \nabla f_{s,t}$ (the gradient with respect to $g_{s,t}$) on $[0,1]^2 \times X$ (see Fig. 7). Let p_0 be an index i critical point of f_0 and q_1 an index i critical point of f_1. Look at the moduli space of flowlines of the vector field from before from p_0 to q_1. This is one-dimensional. We want to consider how these flowlines can break: Compactify the moduli space and consider the resulting boundary. There are four different kinds of boundary components. A flowline can break to a point of index $i+1$ and then go down the point with index i – or it can go to a point of index $i-1$ and then to the index i. Also we can go from s to 0 or s to 1. Define $K : C_*^{\text{Morse}}(X, f_0, g_0) \to C_{*+1}^{\text{Morse}}(X, f_1, g_1)$ counting the ones going from index $*$ to $*+1$. Considering the four kinds boundary components, we obtain $\partial_1 K + K\partial_0 = \phi_0 - \phi_1$. ☐

Claim 16. *The map Φ is an isomorphism.*

Proof. Suppose we have three different choices $(f_0, g_0) \to (f_1, g_1) \to (f_2, g_2)$ and a map $\Phi_{0,1}$: $H_*^{\text{Morse}}(X, f_0, g_0) \to H_*^{\text{Morse}}(X, f_1, g_1)$ and similarly a map $\Phi_{1,2}$. The next step is to prove that $\Phi_{0,2} = \Phi_{1,2} \circ \Phi_{0,1}$. This is done by a similar chain homotopy argument. The last step is to show that $\Phi_{0,0} = \text{id}$ (this is an easy exercise). If we have two different pairs $(f_0, g_0), (f_1, g_1)$, we have $\Phi_{1,0} \circ \Phi_{0,1} = \Phi_{0,0} = \text{id}$ and similarly $\Phi_{0,1} \circ \Phi_{1,0} = \Phi_{1,1} = \text{id}$. This implies that $\Phi_{0,1}$ is an isomorphism. ☐

Remark 17. In order to prove that we really have an invariant, one perhaps should consider a diffeomorphism $\phi : X_0 \to X_1$. In this case, it is not too hard to construct a chain map between the corresponding chain complexes.

2.2.1 Poincaré duality

If X is oriented then $H_*(X) \cong H^{n-*}(X, \mathbb{Z})$. I.e. $H_*^{\text{Morse}}(X, f, g) = H^*_{\text{Morse}}(X, f, g)$ (the Morse cohomology corresponds to counting *upward* flowing flowlines). We have a continuation map Φ : $H_*^{\text{Morse}}(X, f, g) \to H_*^{\text{Morse}}(X, -f, g)$. Now $-f$ has exactly the same critical points but the gradient flows in the opposite direction, so it is almost obvious that $H_*^{\text{Morse}}(X, -f, g) = H^*_{\text{Morse}}(X, f, g)$, and Φ is an isomorphism. It is only almost obvious, because we need to use that X is oriented to get the signs right: Note that $T_pX = U_p \oplus S_p$; given an orientation on X we get an orientation on S_p, and it remains to check that the signs coincide.

There is an analogue of Poincaré duality, using local coefficients, where X is not required to be oriented (or even orientable).

2.2.2 Homology with local coefficients

Let X be a topological space. A local coefficient system of $\mathcal{G}$ on X consists of:

- For each $x \in X$, an abelian group G_x.

- For each path $\gamma : [0,1] \to X$, an isomorphism $\phi_\gamma : G_{\gamma(0)} \to G_{\gamma(1)}$ such that ϕ_γ depends only on the homotopy class of γ (relative to the end points), $\phi_{\text{const}} = \text{id}$ and $\phi_{\gamma_2 \circ \gamma_1} = \phi_{\gamma_2} \circ \phi\gamma_1$ when γ_1 and γ_2 are composable.

For example, the constant local coefficients system, where $G_x = G$ for all x and $\phi_\gamma = \text{id}$. This will correspond to the usual homology with coefficients in G.

Define $H_*(X; \mathcal{G})$ as follows: The chain complex $C_*(X; \mathcal{G})$ is generated by pairs (σ, g) where σ : (simplex or cube) $\to X$, $g \in G_{\text{center}(\sigma)}$ with the relation $(\sigma, g_1 + g_2) = (\sigma, g_1) + (\sigma, g_2)$ and differential

$$\partial(\sigma, g) = \sum_{\sigma' \text{ boundaries of faces}} \pm(\sigma', \phi_\gamma(g))$$

where γ is a path in σ from the center of σ to the center of σ'.

Example 18. Take $X = S^1$. Take $\phi : G_0 \to G_0$ an isomorphism given by going once around the circle, where 0 is some point on the circle. Then $\partial(e_1, g) = (e_0, (1-\phi)g)$. Then $H_1(X; \mathcal{G}) = \text{Ker}(1 - \phi : G \to G)$ and $H_0(X; \mathcal{G}) = G/\text{Im}(1-\phi)$.

Homology with local coefficients is only interesting on non-simply connected spaces. On simply connected spaces, the systems are going to be constant.

Example 19. Take X a smooth manifold with “orientation sheaf” $\mathcal{O}_X$, where

$$\mathcal{O}_X = \mathbb{Z}\{\text{orientations of } T_x X\}/(\overline{\mathcal{O}} = -\mathcal{O})$$

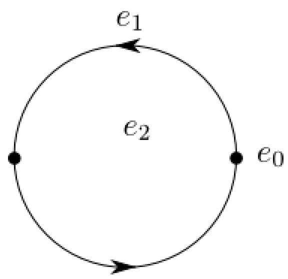

Figure 8: The cells of $\mathbb{R}P^2$.

Example 20. We will compute $H_*(\mathbb{R}P^2; \mathcal{O})$ (see Fig. 8). Going around e_1 we switch the orientation of $\mathbb{R}P^2$. Since we are in the orientation sheaf, signs won't matter, and we get $\partial e_1 = 2e_0$ and $\partial e_2 = 0$. Thus, $H_2(\mathbb{R}P^2; \mathcal{O}) = \mathbb{Z}$, $H_1(\mathbb{R}P^2; \mathcal{O}) = 0$ and $H_0(\mathbb{R}P^2; \mathcal{O}) = \mathbb{Z}_2$.

Theorem 21 (Poincaré duality). *If X is a compact n-dim manifold, then $H_*(X; \mathcal{O}) = H^{n-*}(X; \mathbb{Z})$.*

In the smooth case, the proof is similar as the one in the less general case considered before. Here, we use Morse homology with local coefficients: $C_i^{\text{Morse}}(X, f, g, \mathcal{G})$ is generated by (p, g), where p is an index i critical point and $g \in G_p$ and

$$\partial(p, g) = \sum_{q \text{ index } i-1} \sum_{\gamma \in \check{m}(p,q)} \epsilon(\gamma)(q, \phi_\gamma(g)).$$

Then $H_*^{\text{Morse}}(X, f, g, \mathcal{G}) = H_*(X; \mathcal{G})$. We now proof Poincaré duality exactly as before taking local coefficients in $\mathcal{O}$ in the homology groups.

4th lecture, September 9th

3 Connections in principal bundles

3.1 Vector bundles and Chern classes

Theorem 22. *If X is CW complex, then $H^1(X;\mathbb{Z}) = [X, S^1]$ as sets (and in fact as groups).*

Here, a map $f : X \to S^1$ corresponds to $f^*\alpha$, where $\alpha \in H^1(S^2;\mathbb{Z})$ is a generator.

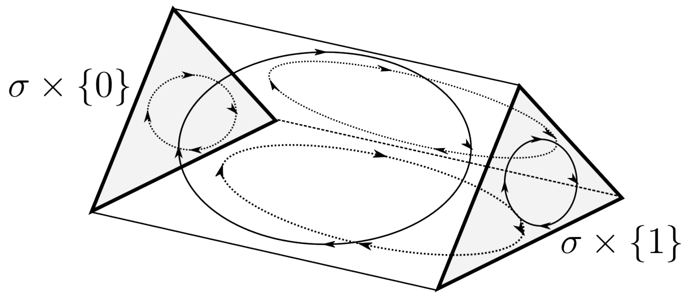

Figure 9: The 2-cell σ.

Proof. We will show that if $f^*\alpha = g^*\alpha$, then f is homotopic to g. Construct a homotopy from f to g by induction on dimension; that is, we want to define $H : [0,1] \times X \to S^1$ such that $H(0,\cdot) = f$, $H(1,\cdot) = g$. We can extend H over $X^0 \times [0,1]$ because S^1 is connected. To extend H over $X^1 \times [0,1]$ consider σ a 1-cell. H is defined on $\{0,1\} \times \sigma \cup [0,1] \times \partial\sigma$. We can define H on the entire square if and only if the square is trivial in S^1, so we have an obstruction[2] in $\pi_1(S^1) = \mathbb{Z}$. The obstructions for the different cells give a cellular cochain $\mathfrak{o} \in C^1_{\text{cell}}(X;\mathbb{Z})$. We claim that $\mathfrak{o}$ is a cocycle; that is, $\partial\mathfrak{o} = 0$. I.e., if σ is any 2-cell, then $\mathfrak{o}(\partial\sigma) = 0$ (see Fig. 9 – the total obstruction is 0). So we have an obstruction $[\mathfrak{o}] \in H^1(X;\mathbb{Z})$. Now, we claim that if $[\mathfrak{o}] = 0$ (that is, $\mathfrak{o} = \delta\eta$), then we can change the definition of H over $X^0 \times [0,1]$ so that it extends over $X^1 \times [0,1]$: When we defined H on the 0-complex, we had a choice of point in S^1, and different choices give different obstruction. Now, if $o = \delta\eta$, η tells us how to choose H on $X^0 \times [0,1]$. Finally, (up to a sign convention) $[\mathfrak{o}] = f^*\alpha - g^*\alpha$: This can be seen by evaluating both sides. We conclude that if $f^*\alpha = g^*\alpha$, then we can extend H over $X^1 \times [0,1]$.

When we try to extend to 2-cells, one cell at a time, we get another obstruction. If σ is a k-cell, the obstruction to extending over $\sigma \times [0,1]$ lives in $\pi_k(S^1) = 0$, when $k > 1$. This proves injectivity. Surjectivity is similar. ☐

We also have the following more general theorem.

Theorem 23. *Let G be an abelian group, and $K(G,n)$ be the corresponding Eilenberg–MacLane space. If X is a CW-complex, then $H^n(X;G) = [X, K(G,n)]$ as sets.*

Now, we will consider $n = 2$.

Definition 24. Let X be a topological space. A *complex line bundle* over X is a space E with a map $\pi : E \to X$ such that $\pi^{-1}(x)$ is a 1-dimensional $\mathbb{C}$-vector space and each $x \in X$ has a neighbourhood U such that $\pi^{-1}(U)$ can be identified with $U \times \mathbb{C}$, and this identification commutes with the two projections to U. If E_1,E_2 are complex line bundles over X, an isomorphism $E_1 \cong E_2$ is a map $E_1 \to E_2$ which commutes with projections and is an isomorphism on each fiber.

[2] For some notes by Hutchings on obstruction theory, see `http://math.berkeley.edu/~hutching/teach/215b-2005/215b.pdf`

Theorem 25. *If X is a CW-complex, then $H^2(X;\mathbb{Z}) = \{$complex line bundles over $X\}/$isomorphism, and the group structure on the right hand side is the tensor product: The map going to the left will be the first Chern class c_1, and we have $c_1(E_1 \otimes E_2) = c_1(E_1) + c_1(E_2)$.*

Lemma 26. *The pullback of E to any cell is trivial.*

Proof of theorem. We will define $c_1(E)$ by obstruction theory. Given E, is it trivial; i.e. is $E \cong X \times \mathbb{C}$? Trivializing is equivalent to finding a non-vanishing section $s : X \to E$. What is the obstruction to finding such a section? We will figure out by defining it cell by cell and seeing what happens. We can construct s over X^0 because $\mathbb{C} \setminus \{0\} \neq \emptyset$. We can now extend s over X^1 by choosing a trivialization by using the lemma, and we can define s since $\mathbb{C} \setminus \{0\}$ is connected. The obstruction to extending over X^2 is a cocycle $o \in C^2_{\text{cell}}(X;\mathbb{Z})$, as $\mathbb{Z} = \pi_1(\mathbb{C} \setminus \{0\})$. Just as before, $\delta\mathfrak{o} = 0$. Now define $c_1(E) = [\mathfrak{o}] \in H^2(X;\mathbb{Z})$. Now E is trivial if and only if $c_1(E) = 0$. If $c_1(E) = 0$, i.e. $\mathfrak{o} = \delta\eta$, then η tells us how to change the choices over the 1-skeleton so that you can extend over 2-skeleton. Once this is done, there are no further obstructions since the higher homotopy groups of $\mathbb{C} \setminus \{0\}$ are 0. □

Definition 27. A rank n complex vector bundle is the same as a line bundle but with fibers $\mathbb{C}^n$ instead of $\mathbb{C}$.

There are Chern classes $c_k(E) \in H^{2k}(X,\mathbb{Z})$, $k = 0,\dots,n$ (where we put $c_0(E) = 1$). Similar to what we did above, $c_k(E)$ is the obstruction to finding $n-k+1$ linearly independent sections over the $2k$-skeleton: If we put $V_{n-k+1}(\mathbb{C}^n) = \{(x_1,\dots,x_{n-k+1}) \in (\mathbb{C}^n)^{n-k+1}\ x_i \text{ linearly independent}\}$, $\pi_{2k-1}(V_{n-k+1}(\mathbb{C}^n)) = \mathbb{Z}$, the lower homotopy groups are 0 and the higher ones are not.

Definition 28. *Chern classes* are defined by the following axioms:

- $c_k(E_1 \oplus E_2) = \sum_{i+j=k} c_i(E_1)c_j(E_2)$ where multiplication is the cup product.
- If E has rank n then $c_n(E) = e(E)$, the Euler class of E (the obstruction to finding a single linearly independent section).
- The Chern class are natural under pullback.

3.2 Connections in vector bundles

Let E be a smooth rank n complex vector bundle over a smooth manifold X. That is, assume all maps in the definition of the vector bundle is smooth. A connection is a way of differentiating sections in a vector bundle:

Definition 29. A *connection* on E is a linear map $\nabla : C^\infty(X;E) \to \Omega^1(X;E)$ from the smooth sections in E to the E-valued 1-forms such that if $f : X \to \mathbb{C}$ and $s \in C^\infty(X;E)$ we have the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s.$$

Now, fix a connection ∇. If $\gamma : [0,1] \to X$ is a smooth path, and if $e \in E_{\gamma(0)}$ then there is a unique section $s \in C^\infty([0,1];\gamma^*E)$ such that $s(0) = e$ and $\nabla s = 0$. The vector $s(1)$ is called the parallel transport of e along γ, and this defines a vector space isomorphism $T_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ (see Fig. 10).

Choose a local trivialization given by linearly independent sections $s_1,\dots,s_n$ over U. So we can write

$$\nabla s_i = \sum_j \omega_{ji} \otimes s_j,$$

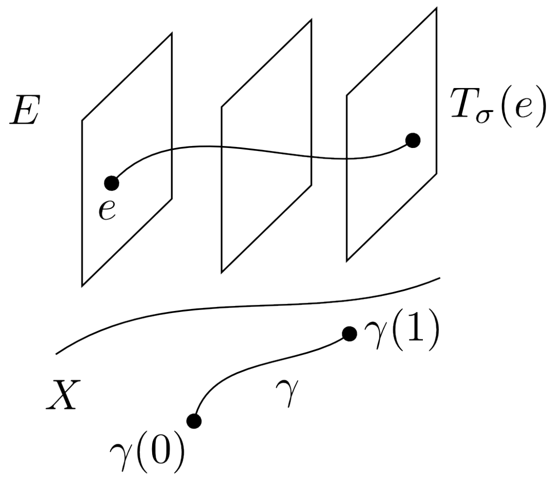

Figure 10: Parallel transport in a vector bundle.

where $\omega_{ji} \in \omega^1(U, \mathbb{C})$. By the Leibniz rule,

$$\begin{aligned}\nabla(\sum_i f_i s_i) &= \sum_i((df_i) \otimes s_i + f_i \nabla s_i) \\ &= \sum_i (df_i \otimes s_i + \sum_j f_i \omega_{ji} \otimes s_j) \\ &= \sum_j (df_j + \sum_i f_i \omega_{ji}) \otimes s_j.\end{aligned}$$

Note that if the ω_{ji} were 0, this would just be ordinary differentiation.

Proposition 30. *The set of connections on E is an affine space over* $\Omega^1(X; End(E))$.

Proof. Fix a connection ∇_0 and let $\nabla : C^\infty(X; E) \to \Omega^1(X; E)$. We claim that ∇ is a connection if and only if $\nabla - \nabla_0$ is in $\Omega^1(X; \mathrm{End}(E))$. We know that $\nabla_0(fs) = df \otimes s + f\nabla_0 s$, and ∇ is a connection if and only if the same holds for ∇, and this is true if and only if

$$(\nabla - \nabla_0)(fs) = f(\nabla - \nabla_0)s,$$

which holds if and only if $\nabla - \nabla_0 \in \Omega^1(X; \mathrm{End}(E))$. $\square$

In a local trivialization $(s_1, \ldots, s_n)$ over U the connection is given by the $\omega \in \Omega^1(U; \mathrm{End}(\mathbb{C}^n))$, the difference between two connections is given by the difference between corresponding ω.

3.3 Curvature of connections in vector bundles

Given a connection ∇, extend it to a map $\nabla : \Omega^k(X; E) \to \Omega^{k+1}(X; E)$ by the Leibniz rule

$$\nabla(\theta \otimes s) = d\theta \otimes s + (-1)^k \theta \wedge \nabla s$$

for $\theta \sin \omega^k(X; \mathbb{C})$, $s \in C^\infty(X; E)$. Note that $\nabla \circ \nabla \neq 0$, unlike the usual covariant derivative.

Claim 31. *The map* $\nabla \circ \nabla : C^\infty(X; E) \to \omega(X; E)$ *is given by multiplication by* $K \in \Omega^2(X; End(E))$

Proof. We need to check that $\nabla \circ \nabla(fs) = f\nabla \circ \nabla(s)$. We have

$$\begin{aligned}\nabla \circ \nabla(fs) &= \nabla(df \otimes s + f\nabla s) \\ &= d(df) \otimes s - df \otimes \nabla s + df \wedge \nabla s + f\nabla\nabla s \\ &= f\nabla\nabla s.\end{aligned}$$

$\square$

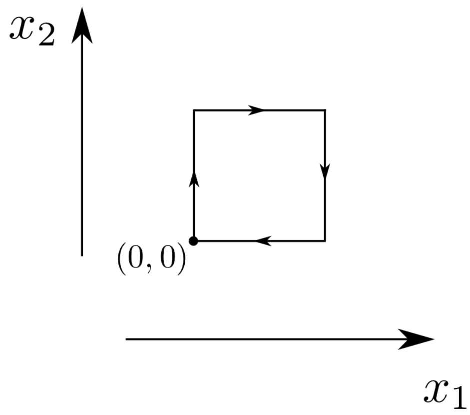

Figure 11: Transporting along a small square γ gives the curvature of a connection.

This K is called the *curvature* of ∇. Its intuitive meaning is given by the following fact: Consider a small square-formed path γ with side length ϵ (see Fig. 11). Then $T_\gamma : E_0 \to E_0$ is given by $T_\gamma = \mathrm{id} + \epsilon^2 K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} + O(\epsilon^4)$. We will see how the curvature is related to the Chern classes.

5th lecture, September 14th 2010

Consider as last time a smooth complex vector bundle $E \to X$. Recall that a *connection* on E is a linear map $\nabla : \Omega^0(X,E) \to \Omega^1(X,E)$ satisfying $\nabla(fs) = df \otimes s + f\nabla s$. That is, it is nothing but a rule for differentiating sections of a vector bundle. Extend this map by the Leibniz rule to a map $\nabla : \Omega^k(X,E) \to \Omega^{k+1}(X,E)$. Consider the composition $\nabla \circ \nabla : \Omega^0(X,E) \to \Omega^2(X,E)$. This is actually multiplication by $K \in \Omega^2(X, \mathrm{End}(E))$ called the *curvature*.

In terms of a local trivialization $s_1, \dots, s_n$ over U. We can write

$$\nabla s_1 = \sum_j \omega_{ji} s_j$$

for $\omega_{ji} \in \Omega^1(U, \mathbb{C})$ or $\omega \in \Omega^1(U, \mathrm{End}(\mathbb{C}^n))$. We see that

$$\begin{aligned}
\nabla(\nabla s_i) &= \nabla(\sum_j \omega_{ji} s_j) \\
&= \sum_j d\omega_{ji} \otimes s_j - \sum_j \omega_{ji} \wedge \nabla s_j \\
&= \sum_j d\omega_{ji} \otimes s_j - \sum_j \omega_{ji} \wedge \sum_k \omega_{kj} s_k \\
&= \sum_j (d\omega_{ji} - \sum_k \omega_{ki} \wedge \omega_{jk}) s_j \\
&= \sum_j (d\omega_{ji} + \sum_k \omega_{jk} \wedge \omega_{ki}) s_j
\end{aligned}$$

We thus see that $K s_i = \sum_j \Omega_{ji} s_j$ where

$$\Omega_{ji} = d\omega_{ji} + \sum_k \omega_{jk} \wedge \omega_{ki}$$

or $\Omega = d\omega + \omega\wedge\omega$ where the $\wedge$ is a combination of $\wedge$ and matrix multiplication. We can also write this as $\Omega = d\omega + \frac{1}{2}[\omega\wedge\omega]$, where

$$[\alpha\wedge\beta](X,Y) = [\alpha(X),\beta(Y)] - [\alpha(Y),\beta(X)],$$

so $[\omega\wedge\omega](X,Y) = 2[\omega(X),\omega(Y)] = 2(\omega\wedge\omega)(X,Y)$. As we saw last time, we can think of curvature in terms of parallel transport, as the parallel transport around a side-length ϵ square is equal to $\mathrm{id} + \epsilon^2 K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}) + O(\epsilon^4)$.

3.4 Chern classes revisited

From now on, assume that X is a compact smooth manifold. Last time we considered the Chern classes $c_k(E) \in H^{2k}(X;\mathbb{Z})$. We want to relate the Chern classes, which measure the global failure of the vector bundle to be trivial to the failure of the connections to be flat. In fact, K vanishes if and only if we locally can choose $s_1,\ldots,s_n$ with $\nabla s_i = 0$.

Denote now the curvature by $\Omega \in \Omega^2(X,\mathrm{End}(E))$. Let $P : \mathrm{End}(\mathbb{C}^n) \to \mathbb{C}$ be a $GL(n,\mathbb{C})$-invariant polynomial, i.e. $P(BAB^{-1}) = P(A)$ if $A \in \mathrm{End}(\mathbb{C}^n)$, $B \in GL(n,\mathbb{C})$. Examples include $P(A) = \sigma_1(A) := \mathrm{tr}(A)$, $P(A) = \sigma_n(A) := \det(A)$ or more generally the k'th elementary symmetric function of the eigenvalues $\sigma_k(A)$,

$$\sigma_k(A) = \sum_{i_1<\cdots<i_n} \lambda_{i_1}\cdots\lambda_{i_k},$$

where λ_i are the eigenvalues of A. In fact, every P is polynomial in the σ_k's. If P is a homogeneous degree k polynomial, define $P(\Omega) \in \Omega^{2k}(X;\mathbb{C})$ by choosing a local trivialization and applying P to Ω. For example $\mathrm{tr}(\Omega) = \sum_i \Omega_{ii}$ and $\det(\Omega) = \sum_\sigma (-1)^\sigma \Omega_{1,\sigma(1)}\cdots\Omega_{n,\sigma(n)}$, where the multiplication is wedge product.

Proposition 32. *$P(\Omega)$ is closed and its cohomology class does not depend on ∇.*

Proof. Closedness: Fix $x \in X$ and choose a local trivialization $s_1,\ldots,s_n$ near x such that $\nabla s_i(x) = 0$: Define the trivialization near x using parallel transport. Then $\Omega = d\omega + \omega\wedge\omega$, so

$$d\Omega = d\omega\wedge\omega - \omega\wedge d\omega.$$

At x, $\omega = 0$ so $d\Omega = 0$. This implies that $dP(\Omega) = 0$ (exercise).

Let ∇_0, ∇_1 be two connections. Consider the pullback of the bundle E to $[0,1]\times X$. On this pullback define $\nabla = (1-t)\nabla_0 + t\nabla_1$. By definition, this is a connection, and $\Omega|_{\{0\}\times X} = \Omega_0$ and $\Omega|_{\{1\}\times X} = \Omega_1$, and $d\Omega = 0$, which implies the second statement. □

Theorem 33. *$[\sigma_k(\Omega)] = (-2\pi i)^k c_k(E)$ in $H^{2k}(X;\mathbb{C})$ (in fact $H^{2k}(X;\mathbb{R})$).*

3.5 Principal bundles

Definition 34. Let G be a Lie group, and let X be a smooth manifold. A *principal G-bundle* is a smooth manifold P, a map $\pi : P \to X$, and a right G-action on P such that for each $x \in X$, there exists a neighborhood U of x and a local trivialization $\pi^{-1}(U) \cong U \times G$ such that the right G-action is multiplication on the right, and the isomorphism commutes with the projections. That is, as a fiber bundle, the fibers are isomorphic to G, and G acts freely and transitively on them.

Example 35. If X is an oriented Riemannian n-manifold, the *frame bundle* is a principal $SO(n)$-bundle defined by

$$P = \{(x,F) \mid x \in X, F : \mathbb{R}^n \overset{\cong}{\to} T_xX \text{ preserving the metric}\}$$

More generally, if $E \to X$ is a complex vector bundle, let $\mathrm{Fr}(E)$ denote the $GL(n,\mathbb{C})$ given by

$$\mathrm{Fr}(E) = \{(x,F) \mid x \in X, F : \mathbb{C}^n \overset{\cong}{\to} E_x\}.$$

Example 36. $G = S^1$, where S^1 is given a metric and an orientation, and the action is given by the circle map. In fact, principal S^1-bundles over X modulo isomorphism are in correspondence to elements of $H^2(X;\mathbb{Z})$, and principal S^1-bundles correspond to complex line bundles in a way we will see later.

If P is a principal G-bundle over X, and if $\rho : G \to \mathrm{Aut}(V)$ is a representation of G, define the *associated vector bundle*

$$P \times_G V = P \times V/(pg, v)\tilde{(}p, gv),$$

where g acts on V by ρ. This is a vector bundle with fiber isomorphic to V. For example, if $P = \mathrm{Fr}(E)$ and $\rho : GL(n,\mathbb{C}) \to \mathrm{Aut}(\mathbb{C}^n)$ is the fundamental representation (given by inclusion), then $P \times_{GL(n,\mathbb{C})} \mathbb{C}^n = E$. If $\rho : GL(n,\mathbb{C}) \to \bigwedge^k \mathbb{C}^n$, then $P \times_{GL(n,\mathbb{C})} \bigwedge^k \mathbb{C}^n = \bigwedge^k E$.

3.6 Connections in principal bundles

Fix a principal G-bundle P over X. Denote the map $P \to X$ by π. For each $\xi \in \mathfrak{g}$, there is a vector field $\sigma_\xi \in \mathrm{Vect}(P)$ tangent to the fiber. In fact, for any $p \in P$, there is a short exact sequence

$$0 \to \mathfrak{g} \to T_pP \overset{\pi_*}{\to} T_{\pi(p)}X \to 0.$$

Now, we want a notion of a "horizontal" direction on P (see Fig. 12). We do this by lifting

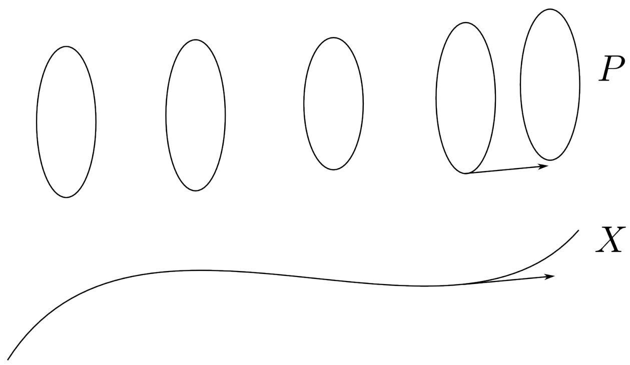

Figure 12: A principal S^1-bundle P over X with a lifted horizontal tangent vector.

tangent vectors of X to tangent vectors of P; in other words, we want a splitting of the above exact sequence.

Definition 37. A *connection* on P is a G-equivariant splitting of this exact sequence. Equivalently, this is a 1-form $A \in \Omega^1(P;\mathfrak{g})$ such that

(1) $A(\sigma_\xi) = \xi$ (that is, on vertical vectors, it does the canonical thing).

(2) If $R_g : P \to P$ is multiplication by $g \in G$, then $R_g^* A = \mathrm{Ad}(g^{-1})A$.

The splitting also gives a horizontal lift $H : T_\pi(p)X \to T_pP$ such that a lift to $T_{pg}P$ is related to the first one by $(R_g)_*$.

We can show that these exist (by sort of patching them together).

Exercise 38. The set of all connections on P is an affine space over $\Omega^1(X; P \times_{\mathrm{Ad}} \mathfrak{g})$.

The vector bundle in this exercise is the infinitesimal automorphism group over the principal bundle: An *automorphism* of P is a map $v : P \to P$ which commutes with the projection to X and the G-action. In a local trivialization, $\pi^{-1}(U) = U \times G$, we have $v(x,g) = (x, f(x)g)$ for a map $f : U \to G$.

Exercise 39. $T_{\mathrm{id}}\mathrm{Aut}(G) = \Omega^0(X; P \times_{\mathrm{Ad}} \mathfrak{g})$.

A connection on P determines a connection on each associated vector bundle $P \times_G V$. How do we define $\nabla s \in \Omega^1(X; P \times_G V)$ for a section s? Locally $s = (p, v)$, where p is a section of of P and $v : X \to V$. Consider the composition

$$T_x X \xrightarrow{p_*} T_{p(x)} P \xrightarrow{A} \mathfrak{g} \xrightarrow{\mathrm{Ad}(\rho)} \mathrm{End}(V),$$

and $\nabla(p, v)$ should be the sum of this and dv. Conversely, starting with a connection in a vector bundle, this gives rise to a principal bundle connection in the frame bundle.

6th lecture, September 16 2010

We begin by recalling a couple of things from last lecture. Consider a principal G-bundle $P \to X$. A connection on P is a G-equivariant smooth choice of splittings of

$$0 \to \mathfrak{g} \to T_p P \xrightarrow{\pi_*} T_{\pi(p)} X \to 0$$

for each $p \in P$. Sometimes we write $\mathfrak{g} = T_{\mathrm{vert}} P = \mathrm{Ker}(\pi_*)$. Equivalently, one could consider horizontal lifts $T_{\pi(p)} X \to T_p P$ (which is comptatible with the principal bundle structure as last time), or a 1-form $A \in \Omega^1(P; \mathfrak{g})$ such that $A|_{T_{\mathrm{vert}} P} \xrightarrow{\cong} \mathfrak{g}$. is the derivative of the G-action and $R_g^* A = \mathrm{Ad}(g^{-1}) \circ A$.

If $\rho : G \to \mathrm{Aut}(V)$ is a representation then A determines a connection ∇_A on the associtated vector bundle $E = P \times_G V$ over X. [Globally, let s be a section of E. Locally we can write $s = [(p, v)]$ on U where $p : U \to P$ and $v : U \to V$. To define a connection on E, we should define $\nabla s \in \Omega^1(U; E)$. To differentiate the p-part, consider

$$TU \xrightarrow{p_*} TP \xrightarrow{A} \mathfrak{g} \xrightarrow{\mathrm{Ad}(\rho)} \mathrm{End}(V) \xrightarrow{\cdot v} V.$$

So do this and add dv.] This is easier to understand in a local trivialization of P, so consider $P|_U = U \times G$ and $E|_U = U \times V$. The connection can be identified with an element $A \in \Omega^1(U; \mathfrak{g})$. Apply $\mathrm{Ad}(\rho)$ to get an element in $\Omega^1(U; \mathrm{End}(V))$ which is our vector bundle connection. Here, $\mathrm{Ad}(\rho) := d\rho_e : \mathfrak{g} \to \mathrm{End}(V)$.

3.7 Curvature in principal bundles

We define curvature by the same formula as in the vector bundle case, $F_A = dA + \frac{1}{2}[A \wedge A] \in \Omega^2(P; \mathfrak{g})$. Because of the equivariance of the connection, it can actually be identified with an element in $\Omega^2(X; P \times_{\mathrm{Ad}} \mathfrak{g})$. Intuitively, we can consider a small square in X and parallel transport along this; doing that we won't get back to the starting point but we will have a shift by a infinitesimal automorphism.

Fact 40. *We have $F_A(V, W) = dA(V_h, W_h)$, where $_h$ denotes the horizontal part of a vector field.*

This is proved directly by a slightly confusing calculation.

Corollary 41. *$F_A \equiv 0$ if and only if the set of horizontal vector fields is an integrable distribution. That is, if V and W are horizontal vector fields, then so is $[V, W]$.*

Proof. Let V and W be horizontal vector fields. Then

$$dA(V, W) = VA(W) - WA(V) - A([V, W]) = -A([V, W]).$$

□

Consider now the special case $G = S^1 = U(1)$, a principal bundle $\pi : P \to X$ and consider a connection $A \in \Omega^1(P; i\mathbb{R})$. If $\theta \in S^1$ then $\theta : P \to P$ satisfies $\theta^A = A$; inputting a vertical vector, this measures how fast the vector is going around the fiber. Now S^1 is abelian, so $F_A = dA \in \Omega^2(P; i\mathbb{R})$. This is related to characteristic classes. Consider a representation $S^1 \to \mathrm{Aut}(\mathbb{C})$. Then $E = P \times_\rho \mathbb{C}$ is a complex line bundle (with a Hermitian metric on the fibers). Observe that $F_A = \pi^*\Omega$, where $\Omega \in \Omega^2(X; i\mathbb{R})$ – this is nothing but the curvature form we considered earlier. To see this, we need to check that F_A is S^1-invariant – which is clear – and that F_A annihilates vertical vectors (which we sort of knew already, but here it is in a different language): Let X be the vector field on P rotating the fibers at unit speed; that is, the derivative of the S^1-action. The Lie derivative of A is $\mathcal{L}_X A = di_X A + i_x dA$, but $\mathcal{L}_X A = 0$, and $di_X A = 0$, so $i_X dA = 0$ which is what we wanted. Furthermore, Ω is closed, as $dF_A = 0$.

Claim 42. $[\Omega] = -2\pi i c_1(E)$.

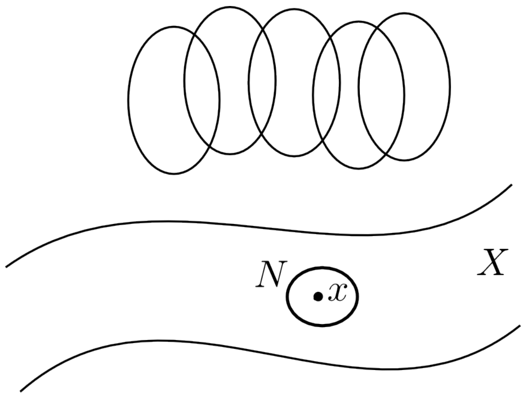

Figure 13: The set $N \subset X$.

Proof. For simplicity assume X is a closed connected oriented surface. Recall that the first Chern class in this case is the obstruction to find one non-vanishing section: Let s be a generic section of E. Then $c_1(E) = \#s^{-1}(0) \cdot \mathrm{PD}(\mathrm{pt}) \in H^2(X; \mathbb{Z})$. We need to check that

$$\int_X \Omega = -2\pi i \cdot \#s^{-1}(0).$$

Away from a small neighborhood N of $s^{-1}(0)$, $p := s/|s|$ is a section of P. We have $\Omega = p^*\pi^*\Omega = p^*F_A$, so

$$\int X\Omega \approx \int_{X\setminus N} \omega = \int_{X\setminus N} p^*F_A = \int_{\partial p(X\setminus N)} A,$$

and this last integral is $-2\pi i \# s^{-1}(0)$. (In $p(X \setminus N)$, see Fig. 13, the boundary goes exactly once around the circle in P (this point is somewhat tricky)). $\square$

4 Spin structures

4.1 Motivation, definition and existence

We like principal bundles, because they give rise to associated vector bundles. For example, the frame bundle $\mathrm{Fr}(X) \to X$ on a Riemannian manifold X is a $SO(n)$-bundle, which, given a representation $\rho : SO(n) \to \mathrm{Aut}(V)$ gives rise to an associated vector bundle $E = \mathrm{Fr}(X) \times_\rho V$. The representations of $SO(n)$ are generated by $\bigwedge^k \mathbb{R}^n$, so the associated bundles are made out of $\bigwedge^k TX$ – these are interesting, as give rise to differential forms, deRham cohomology etc.. We would like to create more associated bundles than these ones which we basically knew already and thereby obtain even more interesting geometry. Recall that $\pi_1 SO(n) = \mathbb{Z}_2$ for $n \geq 2$ (this can be proven by induction using the fibration $SO(n-1) \to SO(n) \to S^n$. In other words, there are representations of $\mathfrak{so}_n$ that do not lift to representations of $SO(n)$. Define $\mathrm{Spin}(n)$ to be the connected double cover of $SO(n)$. There are representations of $\mathfrak{so}_n$ that lift to representations of $\mathrm{Spin}(n)$ but not to representations of $SO(n)$.

Definition 43. A *spin structure* on X is a lift of $\mathrm{Fr}(X)$ to a principal $\mathrm{Spin}(n)$-bundle F over X. That is, we have a 2:1 map $F \to \mathrm{Fr}(X)$ which is compatible with the group actions $\mathrm{Spin}(n) \to F$ and $SO(n) \to \mathrm{Fr}(X)$.

Any representation of $\mathrm{Spin}(n)$ gives you gives you a vector bundle associated to F.

Proposition 44. *There are natural correspondances between the following three sets:*

a) A spin structure on X.

b) An element $\alpha \in H_1(Fr(X); \mathbb{Z}_2)$ that restricts to the generator of each fiber.

c) A trivialization of $Fr(X)$ over the 2-skeleton (assume $n \geq 3$.)

Proof. To go from a) to b), suppose we have a spin structure F on X. Given a loop $\gamma \in \mathrm{Fr}(X)$, lift it to a path $\tilde{\gamma}$ in the double cover F, we let $\alpha(\gamma) = 0$ if and only if $\tilde{\gamma}$ is also a loop. One can think about this physically – I would like to put in the discussion by Hutchings, but it's sort of hard to take notes to what is best illustrated by throwing blackboard erasers around.

Conversely, given α pick a base point $p_0 \in \mathrm{Fr}(X)$. Let F be the set of paths from p_0 to p_1 modulo the equivalence relation that $\gamma \sim \gamma'$ if and only if $\alpha(\gamma - \gamma') = 0$. This defines a spin structure.

To connection between b) and c) is given by obstruction theory arguments. $\square$

From now on we will be sloppy about the metrics on the base space; the space of Riemannian metrics is contractible.

Claim 45. *Let X be a closed oriented manifold. X has a spin structure if and only if $w_2(X) \in H^2(X; \mathbb{Z}_2)$ is 0, where $w_2(X)$ is the Stiefel–Whitney class. If X has a spin structure, then the set of spin structures considered up to isomorphism is an affine space over $H^1(X; \mathbb{Z}_2)$. That is, once we fix a spin structure, the set of spin structures gets identified with $H^1(X; \mathbb{Z}_2)$.*

Proof. We use description b) from the above proposition. The last part of the claim is clear: Given $\alpha \in H^1(\mathrm{Fr}(X); \mathbb{Z}_2)$, restricting to a generator of each fiber, then a $\alpha' \in H^1(\mathrm{Fr}(X), \mathbb{Z}_2)$ also restricts to a generator of each fiber if and only if $\alpha - \alpha' \in H^1(X; \mathbb{Z}_2)$. Concerning the first part, it is at least clear that the obstruction to finding a spin structure is an element in $H^2(X; \mathbb{Z}_2)$; we will not discuss Stiefel–Whitney classes. $\square$

7th lecture, September 21st 2010

4.2 Clifford algebras and spin

Recall that $\mathrm{Spin}(n)$ is defined to be the connected double cover of $SO(n)$. If X is a Riemannian n-manifold, a spin structure on X is a lift $F \to \mathrm{Fr}(X)$ of the $SO(n)$-bundle $\mathrm{Fr}(x)$ to a $\mathrm{Spin}(n)$-bundle F.

We observe that $\mathrm{Spin}(2) \cong U(1)$. Also $\mathrm{Spin}(3) \cong SU(2)$, as $SU(2)$ acts on $\mathbb{C}P^1 \approx S^2 \subseteq \mathbb{R}^3$ – this gives a map $SU(2) \to SO(3)$. It is a double cover, because $SU(2) \cong S^3$; note also that both $-I$ and I in $SU(2)$ acts as the identity on $\mathbb{C}P^1$.

It is also true that $\mathrm{Spin}(4) \cong SU(2) \times SU(2)$. For $n > 4$ we don't have these nice "coincidences".

Let V be an n-dimensional real vector space with an inner product. Define the *Clifford algebra* $\mathrm{Cl}(V)$ to be the tensor algebra of V modulo the relation $ab + ba = -2\langle a, b\rangle$ for $a, b \in V$. Denote by $\mathrm{Cl}(n)$ the Clifford algebra $\mathrm{Cl}(\mathbb{R}^n)$.

Let us consider a few examples. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. We have

$$\mathrm{Cl}(1) = \mathbb{R}\{1, e_1\}/(e_1^2 = -1) = \mathbb{C}.$$

Similarly,

$$\mathrm{Cl}(2) = \mathbb{R}\{1, e_1, e_2, e_1e_2\}/(e_1^2 = e_2^2 = -1, e_1e_2 = -e_1e_2) = \mathbb{H}$$

by identifying $e_1 = i, e_2 = j, e_1e_2 = k$.

Proposition 46. *There is a canonical isomorphism of vector spaces $Cl(V) \cong \bigwedge^* V$ (but not an isomorphism of algebras). In particular* $\dim Cl(V) \cong 2^n$.

Proof. Define $\phi : \bigwedge^* V \to \mathrm{Cl}(V)$ by

$$\phi(a_1 \wedge \cdots \wedge a_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma a_{\sigma(1)} \cdots a_{\sigma(k)}.$$

Note that this is a well-defined linear map; changing the order of the tensors on the left hand side, the sign changes accordingly. We claim that ϕ is an isomorphism. Observe that $\mathrm{Cl}(V)$ is filtered by word length. That is, let $\mathrm{Cl}_k(V)$ be the subspace of $\mathrm{Cl}(V)$ generated by words of length less than or equal to k, the word 1 having length 0. Note that $\mathrm{Cl}_0(V) = \mathbb{R}$. Multiplication preserves the filtration, $\mathrm{Cl}_k(V) \cdot \mathrm{Cl}_l(V) \subseteq_{k+1} (V)$. As a result, there is an associated graded algebra

$$G\mathrm{Cl}(V) = \bigoplus_k \frac{\mathrm{Cl}_k(V)}{\mathrm{Cl}_{k+1}(V)}.$$

By the above there is a welldefined well-defined multiplication $G\mathrm{Cl}(V) \otimes G\mathrm{Cl}(V) \to G\mathrm{Cl}(V)$ given by multiplying things together and ignoring words of shorter length. We have an algebra isomorphism $G\mathrm{Cl}(V) = T(V)/(ab + ba = 0) = \bigwedge^* V$. This implies that $\mathrm{Cl}(V)$ and $\bigwedge^* V$ have the same dimension, and ϕ induces this (details left to the audience). □

We can consider Clifford multiplication as exterior multiplication plus an extra term (as the inner product tends to zero, Clifford multiplication tends to exterior multiplication).

Proposition 47. *Under the canonical vector space isomorphism $Cl(V) \cong \bigwedge^* V$, if $a \in V$, $b \bigwedge^*(V)$ then Clifford multiplication $a \cdot b = a \wedge b - a \lrcorner b$, where we consider $b \in \bigwedge^*(V)$ as an element of the dual.*

Proof. Check this on an orthonormal basis $e_1, \dots, e_n$. □

Suppose $n \geq 2$, $a, b \in V$ with $a \neq 0$. In $\mathrm{Cl}(V)$,

$$\begin{aligned} aba^{-1} &= -\frac{aba}{\langle a, a\rangle} = -\frac{a}{\langle a, a\rangle}(-ab - 2\langle a, b\rangle) = -\left(b - 2\frac{\langle a, b\rangle}{\langle a, a\rangle}a\right) \\ &= -(\text{reflection of } b \text{ across } a^\perp). \end{aligned}$$

Now *define* $\mathrm{Spin}(n) = \{x \in \mathrm{Cl}(n) \mid x = a_1 \cdots a_{2k}, a_i \in \mathbb{R}^n, \|a_i\| = 1\}$. We have a map $\pi : \mathrm{Spin}(n) \to SO(n)$ given by $x \mapsto (y \mapsto xyx^{-1})$ which by the above is a product of reflections across $a_1^\perp, \dots, a_{2k}^\perp$.

Claim 48. *Spin(n) as defined above is a connected double cover of $SO(n)$.*

Proof. First, the map π is surjective, because every element of $SO(n)$ is the product of an even number of reflections. Writing $\mathbb{Z}_2 = \{1, -1\} \subseteq \ker(\pi)$, and in fact we have equality, which can be seen by expanding in an orthonormal basis. It remains to see that the cover is connected and not just say $SO(n) \times \mathbb{Z}_2$. We simply find a path from 1 to -1; for example

$$t \mapsto e_1((\cos \pi t)e_1 + (\sin \pi t)e_2),$$

so $t = 0$ gives $e_1e_1 = -1$ and $t = 1$ gives $e_1(-e_1) = 1$. □

The definition of Spin(n) in terms of the Clifford algebra makes it easier to describe the representation theory of Spin(n), as the representations of Spin(n) come from representations of Cl(n).

Consider the first "basic" representations of Cl(n): For $n = 2$, we have a representation on $\mathbb{C}^2$ by $e_1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $e_2 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ (note that $e_1^2 = e_2^2 = -1$ and $e_1 e_2 = -e_2 e_1$.

For $n = 3$, we have another representation on $\mathbb{C}^2$ where $e_i \mapsto \sigma_i$, the Pauli matrices

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

For $n = 4$, we represent on $\mathbb{C}^4$ by block matrices

$$e_0 \mapsto \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, e_i \mapsto \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}.$$

Exercise 49. Under appropriate conventions for identifying Spin$(3) \cong SU(2)$, the above representation of Cl(3) gives the fundamental representation of $SU(2)$ on $\mathbb{C}^2$.

Let V be a representation of Cl(n). This restricts to a representation of Spin(n). If $F \to X$ is a spin structure, we get an associated vector bundle $S := F \times_{\mathrm{Spin}(n)} V \to X$.

Claim 50. *S is a module of the bundle of Clifford algebras $Cl(TX) \to X$. That is, there is a bundle map $cl : TX \to End(S)$ satisfying $cl(a)cl(b) + cl(b)cl(a) = -2\langle a, b\rangle I$.*

Proof. Because F is a lift of the frame bundle, we get $TX = F \times_{\mathrm{Spin}(n)} \mathbb{R}^n$, where Spin$(n)$ acts on $\mathbb{R}^n$ by projecting to $SO(n)$. We have $\mathrm{Cl}(TX) = F \times_{\mathrm{Spin}(n)} \mathrm{Cl}(n)$, where Spin$(n)$ acts on Cl(n) by conjugation, and $S = F \times_{\mathrm{Spin}(n)} V$, so we can define a map $\mathrm{Cl}(TX) \otimes S \to S$ by

$$(p, a) \otimes (p, v) \to (p, av)$$

for $p \in F, a \in \mathrm{Cl}(n), v \in V$. We need to check that this is welldefined; that is, that if $g \in \mathrm{Spin}(n)$ then $(pg, g^{-1}ag) \otimes (pg, g^{-1}v) \mapsto (pg, g^{-1}av)$. This is clear as $(g^{-1}ag)(g^{-1}v) = g^{-1}av$. □

If V is the "basic" representation of Cl(n) discussed before, S is called the *spin(or) bundle* ssociated to the spin structure.

If $V = \mathrm{Cl}(n) \cong \bigwedge^* \mathbb{R}^n$ given by multiplication, then $S = \bigwedge^* TX$. Here $\mathrm{cl}(a)b = a \wedge b - a \lrcorner b$, $a \in TX, b \in \bigwedge^* TX$.

Another motivation for the Clifford algebra is the following. Consider $\Delta = \sum_{i=0}^n -\frac{\partial^2}{\partial x_i^2}$ on $C^\infty(\mathbb{R}^n, \mathbb{R})$. Dirac wanted to find the square root of this operator. We could try writing $D = \gamma_1 \frac{\partial}{\partial x_1} + \cdots + \gamma_i \frac{\partial}{\partial x_n}$. Then

$$D^2 = \sum_i \gamma_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i<j} (\gamma_i \gamma_j + \gamma_j \gamma_i) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}.$$

We thus want $\gamma_i^2 = -1$ and $\gamma_i \gamma_j + \gamma_j \gamma_i = 0$ for $i \neq j$. These are just the Clifford relations. We can't have them satisfied for complex or real numbers, but it is possible to do for matrices; instead we consider a representation $e_i \mapsto \gamma_i$ of Cl(n).

Fact 51. *Let $F \to X$ be a spin structure, let $S \to X$ be the spin bundle. Then there is a canonical connection ∇ on S, there is a well-defined operator $D : C^\infty(X, S) \to C^\infty(X, S)$ called the* Dirac operator *satisfying $D^2 = \nabla^* \nabla$ plus lower order terms involving curvature.*

We will discuss this fact in detail next time.

8th lecture, September 23rd 2010

Let X be a Riemannian manifold X with a spin structure $F \to \mathrm{Fr}(X) \to X$. Last time we described the Clifford algebra $\mathrm{Cl}(n)$ and $\mathrm{Spin}(n) \subset \mathrm{Cl}(n)^x$. We also described "basic representations" V of $\mathrm{Cl}(n)$; call them "spin representations" from now on. Restricting any representation $\rho : \mathrm{Cl}(n) \to \mathrm{End}(V)$ to $\rho : \mathrm{Spin}(n) \to \mathrm{Aut}(V)$, we get an associated vector bundle $S = F \times_\rho V$. Now S is a left module over $\mathrm{Cl}(TX)$, $\mathrm{cl} : TX \otimes S \to TX \otimes S$,

$$\mathrm{cl}(v)\mathrm{cl}(w) + \mathrm{cl}(w)\mathrm{cl}(v) = -2\langle v, w\rangle \mathrm{id}_S.$$

Notice that $TX = F \times_{\mathrm{Spin}(n)} \mathbb{R}^n$ and $\mathrm{Cl}(TX) = F \times_{\mathrm{Spin}(n)} \mathrm{Cl}(n)$, where $\mathrm{Spin}(n)$ acts on $\mathrm{Cl}(n)$ by conjugation, so we get a map $\mathrm{Cl}(TX) \otimes S \to S$; given $p \in F$, we map $(p, x) \otimes (p, v) \mapsto (p, xv)$. This is well-defined since for $g \in \mathrm{Spin}(n)$, we have

$$(pg, g^{-1}xg) \otimes (pg, g^{-1}v) \otimes (pg, g^{-1}xv).$$

Example 52. If $V \cong \mathrm{Cl}(n) \cong \bigwedge^* \mathbb{R}^n$, then $S = \bigwedge^* T^*X$. If $v \in T^*X$, $\alpha \in \bigwedge^* T^*X$, then

$$\mathrm{cl}(v)\alpha = v \wedge \alpha - v \lrcorner \alpha.$$

If S is the vector bundle associated to a unitary representation of $\mathrm{Cl}(n)$, then there is a canonical Hermitian connection on S. Here a connection on a Hermitian vector bundle E is called *Hermitian/unitary* if for sections s_1, s_2 of E, then $d\langle s_1, s_2\rangle = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle$ – this is equivalent to parallel transport being unitary (exercise: check this; the idea is to parallel transport along small paths). The connection A on S is the Levi–Civita connection on TX or $\mathrm{Fr}(X)$. Here, the working class definition of the Levi–Civita connection is the unique connection with the following property: We can choose a coordinate chart with $g_{ij}(x) = \delta_{ij} + O(|x|^2)$ and at 0, we have $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$.

We have $A \in \Omega^1(\mathrm{Fr}(X), \mathfrak{so}_n)$, and $\mathfrak{spin}_n = \mathfrak{so}_n$, and if we denote by π the covering $F \to \mathrm{Fr}(X)$, we have $\pi^* A \in \Omega^1(F, \mathfrak{spin}_n)$. Thus we have a connection ∇ on the associated spin bundle.

Proposition 53. *The connection has the following properties:*

1) *It is unitary.*

2) *It is a "module derivation": If α is a section of TX, and ψ is a section of S, then*

$$\nabla(\alpha \cdot \psi) = (\nabla\alpha) \cdot \psi + \alpha \cdot \nabla\psi.$$

Here the second ∇ is the Levi–Civita connection.

Proof of property 2). Choose a local trivialization of F over U, so $F|_U \cong U \times \mathrm{Spin}(n)$ and $A \in \Omega^1(U; \mathfrak{spin}(n))$. We have two maps $\mu : \mathrm{Spin}(n) \to \mathrm{Aut}(\mathbb{R}^n)$ given by the composition of the projection to $SO(n)$ and the fundamental representation, and $\rho : \mathrm{Spin}(n) \to \mathrm{Aut}(V)$. Now $\alpha : U \to \mathbb{R}^n$, $\psi : U \to V$. We have $\nabla\psi = (d + \rho_* \circ A)\psi$ and $\nabla\alpha = (d + \mu_* \circ A)\alpha$. Also $\nabla(\alpha \cdot \psi) = (d + \rho_* \circ A)(\alpha \cdot \psi)$. By the Leibniz rule, we need to have

$$(\rho_* \circ A)(\alpha dot \psi) = (\mu_* \circ A)(\alpha) \cdot \psi + \alpha \cdot (\rho_* \circ A)(\psi).$$

Observe that if $g \in \mathrm{Spin}(n)$ then differentiating the equation $g(\alpha \cdot \psi) = (g\alpha g^{-1}) \cdot (g\psi)$ with respect to g at the identity, we get for $A \in \mathfrak{spin}(n)$

$$\rho_*(A)(\alpha \cdot \psi) = (\mu_* A)(\alpha) \cdot \psi + \alpha \cdot \rho_*(A)(\psi).$$

□

4.3 The Dirac operator

The Dirac operator is an operator $D : \Gamma(S) \to \Gamma(S)$ given by the composition $\Gamma(s) \xrightarrow{\nabla} \gamma(T^*X \otimes S) \xrightarrow{\mathrm{cl}} \Gamma(S)$. If e_i is a orthonormal frame then

$$D\psi = \sum_i e_i \cdot \nabla_{e_i}\psi.$$

Example 54. Let us consider the Dirac operator on $\mathbb{R}$. Then $\mathrm{cl}(e_1) = (i)$, $S = \mathbb{R} \times \mathbb{C}$ and $D\psi = i\partial_1\psi$.

On $\mathbb{R}^2$ we have $\mathrm{cl}(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\mathrm{cl}(e_2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Now $S = \mathbb{R}^2 \times \mathbb{C}^2$. Writing $\psi = (\psi_1, \psi_2)$ we obtain

$$\begin{aligned} D\psi &= \mathrm{cl}(e_1)\partial_1\psi + \mathrm{cl}(e_2)\partial_2\psi \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \partial_1\psi_1 \\ \partial_1\psi_2 \end{pmatrix} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} \partial_2\psi_1 \\ \partial_2\psi_2 \end{pmatrix} \\ &= \begin{pmatrix} -\partial_1\psi_2 + i\partial_2\psi_2 \\ \partial_1\psi_1 + i\partial_2\psi_1 \end{pmatrix} = \begin{pmatrix} -2\frac{\partial\psi_2}{\partial z} \\ 2\frac{\partial\psi_1}{\partial \bar{z}} \end{pmatrix}, \end{aligned}$$

and

$$D = \begin{pmatrix} 0 & -2\frac{\partial}{\partial z} \\ 2\frac{\partial}{\partial \bar{z}} & 0 \end{pmatrix}.$$

From these examples it is clear that the Dirac operator is a square root of the Laplacian.

Example 55. If X is a Riemannian manifold, and $V = \mathrm{Cl}(n) \cong \bigwedge^* T^*(X)$, and $D : \Omega^*X \to \Omega^*X$. Then $D = d + d^*$ where (depending on sign convention) we have $d^* = \pm * d*$: If $e_1, \dots, e_n$ is an orthonormal frame,

$$d\alpha = \sum_i e_i \wedge \nabla_i\alpha, \quad d^*\alpha = -\sum e_i \lrcorner \nabla_i\alpha,$$

and the claim about D follows from previous observations. The square of this is $D^2 = dd^* + d^*d$ which is known as the "Hodge Laplacian".

On $\mathbb{R}^n$, $D = \sum_i e_i \frac{\partial}{\partial x_i}$ and $D^2 = \sum_i \frac{\partial^2}{\partial x_i^2} = \Delta$, the Laplacian. The goal now is to see that on a general Riemannian manifold $D^2 = \nabla^*\nabla + \frac{s}{4}$. The term $\nabla^*\nabla$ is known as the "connection Laplacian", and s is the scalar curvature.

4.4 Review of differential operators

Let E, F be (Hermitian) vector bundles over a (Riemannian) n-manifold X. A *differential operator* $D : \Gamma(E) \to \Gamma(F)$ is a linear map such that in local coordinates, $D = \sum_{|\alpha| \le m} A_\alpha D_\alpha$, where $\alpha = (i_1, \dots, i_k)$ is an ordered list of indices, $i_1, \dots, i_k \in \{1, \dots, n\}$, and $D_\alpha = \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}}$ and $A_\alpha \in \mathrm{Hom}(E, F)$. It is said be of order m if $A_\alpha \neq 0$ for some α with $|\alpha| = m$.

Example 56. A differential operator of order 0 is a bundle map.
Covariant derivative $\nabla : \Gamma(E) \to \Gamma(T^*X \otimes E)$ is a differential operator of order 1. The Dirac operator $D : \Gamma(S) \to \Gamma(S)$ is also a differential operator of order 1. The exterior derivative $d : \Gamma(\bigwedge^k T^*X) \to \Gamma(\bigwedge^{k+1} T^*X)$ is also of order 1.

When assuming that the manifold X is Riemannian and the bundle is Hermitian, we can form a *formal adjoint* $D^* : \Gamma(F) \to \Gamma(E)$ characterized by

$$\int_X \langle D\alpha, \beta\rangle, \mathrm{dvol} = \int_X \langle \alpha, D^*, \beta\rangle \mathrm{dvol}$$

for sections α, β of E, F that are compactly supported on $\int(X)$. Then D^* is also a differential operator is also an operator of order m (as can be seen by integration by parts).

If D is a differential operator of order m, its terms with $|\alpha| = m$ determine a bundle map $\sigma(D) : \mathrm{Sym}^m(T^*X) \otimes E \to F$ called the ***symbol*** of the differential operator. Usually $\sigma(D)$ is defined to be $i^m \cdots$ terms of order m.

Exercise 57. This does not depend on the coordinates. The idea is, that if we write the differential operator in new coordinates, we need the chain rule to do the differentiation, and the highest order terms survive.

Example 58. A connection on E is equivalent to a first order differential operator $\nabla : \Gamma(E) \to \Gamma(T^*X \otimes E)$ such that $\sigma(D) : T^*X \otimes E \to T^*X \otimes E$ is the identity.

Example 59. The Dirac operator $D : \Gamma(S) \to \Gamma(S)$ has symbol $\sigma(D) : T^*X \otimes S \to S$ is cl.

9th lecture, September 28 2010

4.5 Elliptic operators

Last time we considered differential operators. If we have two complex vector bundles $E \to X$, $F \to X$, a map $D : \Gamma(E) \to \Gamma(F)$ is a differential operator of order m, if locally

$$D = \sum_{|\alpha| \leq m} A_\alpha D_\alpha,$$

where $\alpha = (i_1, \ldots, i_k)$, $D_\alpha = \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}}$, where $A_\alpha \in \Gamma(\mathrm{Hom}(E, F))$ and $A_\alpha \neq 0$ for some α (at each $x \in X$). We define the symbol

$$\sigma(D) = i^m \sum_{|\alpha| = m} A_\alpha D_\alpha,$$

which is a section of $\mathrm{Hom}(\mathrm{Sym}^m T^*X \otimes E, F) = \mathrm{Hom}(\mathrm{Sym}^m T^*X, \mathrm{Hom}(E, F))$. See also the exercise above.

Definition 60. A differential operator D is *elliptic* if for $\xi \in T_x^*X \setminus \{0\}$ the map $\sigma(D)(\xi, \ldots, \xi) : E_x \to F_x$ is an invertible linear map.

Example 61. The Cauchy–Riemann operator

$$\overline{\partial} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} : C^\infty(\mathbb{C}, \mathbb{C}) \to C^\infty(\mathbb{C}, \mathbb{C})$$

is a first order operator on the trivial line bundle with symbol

$$\sigma(\overline{\partial})(a\, dx + b\, dy) = i(a + bi)$$

for $a, b \in \mathbb{R}$.

We have the following vague intuition about elliptic operators: An elliptic operator "remembers" the m'th derivative.

Fact 62. *If D is an order m elliptic operator, and if $D\psi$ is k times differentiable, then ψ is $k+m$ times differentiable.*

If ψ is supported in a compact set K, then

$$\|\psi\|_{L^2_{m+k}} \leq C_K(\|\psi\|_{L^2_k} + \|D\psi\|_{L^2_k})$$

in Sobolov spaces, which we will get back to later.

Example 63. The Laplacian

$$\Delta = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2} : C^\infty(\mathbb{R}^n, \mathbb{C}) \to C^\infty(\mathbb{R}^n, \mathbb{C})$$

is a differential operator. If $\xi = a_1\,dx_1 + \cdots + a_n\,dx_n$, we have

$$\sigma(\Delta)(\xi, \xi) = a_1^2 + \cdots + a_n^2$$

so Δ is elliptic.

Let X be a (compact) n-dimensional Riemannian oriented manifold, $d : \Omega^k(X) \to \Omega^{k+1}(X)$ and formal adjoint $d^* : \Omega^{k+1}(X) \to \Omega^k(X)$.

Exercise 64. We have $d^* = \pm d * d$ where $* : \Omega^k(X) \to \Omega^{n-k}(X)$ is given as follows: If $e_1, \ldots, e_n$ is an orthonormal basis for T_x^*X, then

$$*(e_1 \wedge \cdots \wedge e_k) = e_{k+1} \wedge \cdots \wedge e_k.$$

Claim 65. *The operator $d + d^* : \Omega^*(X) \to \Omega^*(X)$ is elliptic.*

Proof. Let $e_1, \ldots, e_n$ be an oriented orthonormal basis for T_x^*X. We have

$$\sigma(d)(e_1) = ie_1\wedge,$$

and

$$\sigma(d^*)(e_1) = -ie_1\lrcorner,$$

so

$$\sigma(d + d^*)(e_1) = i(e_1 \wedge -e_1\lrcorner).$$

This gives a bijection between basis vectors containing e_1 and basis vectors not containing e_1. The symbol is homogeneous of degree 1, so it is invertible. □

If X has a spin structure and if S is a spin bundle on X (coming from a representation of $\mathrm{Cl}(n)$), then the Dirac operator $D : \Gamma(S) \to \Gamma(S)$ is also elliptic. The operator $d + d^*$ is an example of a Dirac operator coming from the representation of $\mathrm{Cl}(n)$ on itself acting by multiplication. Recall that the Dirac operator is the composition

$$\Gamma(S) \xrightarrow{\nabla} \Gamma(T^*X \otimes S) \xrightarrow{d} \Gamma(S).$$

In fact, in general the symbol of a composition is the composition of symbols, so $\sigma(D) = \sigma(d) \circ \sigma(\nabla)$. Here $\sigma(\nabla) = i \cdot \mathrm{id}$, and d is a 0 order operator, and $\sigma(d) = d$, so $\sigma(D) = i \cdot d \in \mathrm{Hom}(T^*X, \mathrm{Hom}(S, S))$. If $\xi \in T^*X \setminus \{0\}$, then $d(\xi)$ is invertible because $\mathrm{cl}(\xi)^2 = -|\xi|^2\mathrm{id}_S$, and the Dirac operator is elliptic.

Fact 66 (Awesome facts about elliptic operators)**.** *Suppose X is compact and Riemannian with Hermitian vector bundles E, F. Let $D : \Gamma(E) \to \Gamma(F)$ be an elliptic operator. Then*

- $\ker(D)$ *is finite dimensional*
- $\Gamma(F) = Im(D) \oplus \ker(D^*)$ *(where the two factors are L^2-orthogonal).*
- *The index $ind(D) = \dim(\ker(D)) - \dim(\ker(D^*))$ is invariant under deformation of D through elliptic operators. In fact, $ind(D)$ is determined by the homotopy class of $\sigma(D) : T^*X \setminus \{0\text{-section}\} \to Hom^x(E, F)$, and there is a topological formula for the index (Atiyah–Singer Index theorem). Here Hom^x denotes invertible bundle maps.*

Assume again that X is compact and Riemannian. We will consider the second of these and see why it is powerful. Consider $D = d + d^*$. A differential form is called *harmonic* if $d\alpha = d^*\alpha = 0$. Let $\mathcal{H}_k$ be the set of harmonic forms of degree k.

Theorem 67 (Hodge decomposition)**.** *There is an L^2-orthogonal decomposition*

$$\Omega^k(X) = \mathcal{H}_k \oplus d\Omega^{k-1} \oplus d^*\Omega^{k+1}.$$

Proof. Consider $d + d^* : \Omega^{\text{even}}(X) \to \Omega^{\text{odd}}(X)$ (or vice-versa). This is elliptic. Observe that $\mathcal{H}_k$, $d\Omega^{k-1}$, and $d^*\Omega^{k+1}$ are mutually orthogonal (using the adjoint property). So we need to prove that if $\alpha \perp \mathcal{H}_k$, then $\alpha \in d\Omega^{k-1} \oplus d^*\Omega^{k+1}$. The map $d+d^*$ is self-adjoint, so $\Omega^{\text{even}} = \mathcal{H}_{\text{even}} \oplus (d+d^*)\Omega^{\text{odd}}$ using the property above. We need to show that $(d + d^*)\Omega^{\text{odd}} = d\Omega^{\text{odd}} \oplus d^*\Omega^{\text{odd}}$. The inclusion from left to right is obvious, and the one from right to left follows from the fact that both $d\Omega^{\text{odd}}$ and $d^*\Omega^{\text{odd}}$ are orthogonal to $\mathcal{H}_{\text{even}}$. □

Corollary 68. *Any deRham cohomology class has a unique harmonic representative.*

Proof. Any closed form in $\Omega^k(X)$ is in $\mathcal{H}_k \oplus d\Omega^{k-1}$, and there is unique form in $\mathcal{H}_k$ representing the class. □

This implies that $\dim \mathcal{H}_k = \dim H^*(X; \mathbb{R})$, so $\operatorname{ind}(d + d^* : \Omega^{\text{even}} \to \Omega^{\text{odd}}) = \chi(X)$. (Note that the index of $d + d^* : \Omega^*(X) \to \Omega^*(X)$ is 0, since the map is selfadjoint.)

We turn now to the Bochner–Weitzenböck formula. Consider the Dirac operator $D : \Gamma(S) \to \Gamma(S)$. Then $D^2 = \nabla^*\nabla + \frac{s}{4}\mathrm{id}_S$, where $\nabla^*\nabla$ is called the "connection Laplacian", and s is scalar curvature, $s = \sum_{i,j} R_{ijij}$. If ∇^{LC} is the Levi–Civita connection, in local coordinates the $dx^i dx^j$ component of the curvature (of any connection) is the commutator $[\nabla_i, \nabla_j]$, where $\nabla_i = \nabla_{\frac{\partial}{\partial x_i}}$, and we have

$$R_{ijkl} = \langle [\nabla_i, \nabla_j] e_l, e_k \rangle.$$

Exercise 69. The unit n-sphere has scalar curvature $n^2 - n$.

Note that $\Delta = d^*d$ on functions. (More generally one could consider the "Hodge Laplacian" $dd^* + d^*d$ on forms.) We can therefore consider D^2 as an analogue of this on vector bundles.

10th lecture, September 30th 2010

Let X be a Riemannian n-manifold with a spin structure, V a representation of $\mathrm{Cl}(n)$, and let S be the associated spin bundle. Then we have the Dirac operator $D : \Gamma(S) \to \Gamma(S)$ defined by the composition $\Gamma(S) \xrightarrow{\nabla} \Gamma(T^*X \otimes S) \xrightarrow{\text{cl}} \Gamma(S)$. If $e_1, \dots, e_n$ is an orthonormal frame at a point $x \in X$, then $D\psi(x) = \sum_i e_i \cdot \nabla_{e_i}\psi$.

Theorem 70 (The Bochner–Lichnerowicz–Weitzenböck formula)**.** *The following equation holds.*

$$D^2 = \nabla^*\nabla + \frac{s}{4}.$$

Proof. The formula relates to section of S, so we just have to prove it at a point. Choose local coordinates centered at a point so that $g_{ij}(x) = \delta_{ij} + O(|x|^2)$. Write $e_i = \frac{\partial}{\partial x_i}$. Then $\nabla_{e_i} e_j(0) = 0$. Write $\nabla_i = \nabla_{e_i}$. At our point, we have

$$\begin{aligned} D^2\psi &= e_i \cdot \nabla_i(e_j \cdot \nabla_j \psi) \\ &= e_i \cdot ((\nabla_i e_j) \cdot \nabla_j \psi + e_j \cdot \nabla_i \nabla_j \psi) \\ &= e_i e_j \nabla_i \nabla_j \psi \\ &= -\nabla_i \nabla_i \psi + \frac{1}{2} \sum_{i \neq j} e_i e_j [\nabla_i, \nabla_j] \psi \end{aligned}$$

using the summation convention and the facts $e_ie_i = -1$ and $e_ie_j = -e_je_i$. Note that $[\nabla_i, \nabla_j]$ is the $dx_i\, dx_j$ component of curvature. We claim that $\nabla^*\nabla = -\nabla_i\nabla_i$ and $\frac{1}{2}\sum_{i\neq j} e_ie_j[\nabla_i, \nabla_j] = \frac{s}{4}$.

In our point, $\nabla\psi = (\nabla_i\psi)\, dx_i$. If α, β are compactly supported sections on $\int(X)$, α a section of S and β a section of $T^*S \otimes S$ then

$$\int_X \langle \nabla\alpha, \beta\rangle d\mathrm{vol} = \int_X \langle \alpha, \nabla^*\beta\rangle d\mathrm{vol}.$$

On $\mathbb{R}^n$ we write $\beta = \beta_i\, dx_i$, and we get

$$\begin{aligned}\int_{\mathbb{R}^n} \langle \nabla\alpha, \beta\rangle d\mathrm{vol} &= \int_{\mathbb{R}^n} \langle \nabla_i\alpha, \beta_j\rangle g^{ij} d\mathrm{vol} \\ &= \int_{\mathbb{R}^n} (\partial_i\langle \alpha_i, \beta_j\rangle - \langle \alpha, \nabla_i\beta_j\rangle) g^{ij} d\mathrm{vol}.\end{aligned}$$

We would like to integrate by parts, which would be possible if not for the $g^{ij}d$vol. Instead we get

$$\nabla^*(\beta_j\, dx_j) = -\nabla_j\beta_j + \text{error term from derivatives of metric.}$$

At the origin though, this error term vanishes, and $\nabla^*(\beta_j\, dx_j) = -\nabla_j\beta_j$, which proves the first claim.

For the second claim, as before $[\nabla_i, \nabla_j]$ is the $dx_i\, dx_j$ component of F_∇. The question is what the relation between the scalar curvature (and the Levi–Civita connection) and our connection ∇. We claim that $[\nabla_i, \nabla_j] = -\frac{1}{4}R_{ijkl}e_ke_l$, where R is the Riemann curvature tensor. We have a map of Lie algebras $\pi : \mathfrak{spin}(n) \to \mathfrak{so}(n)$ given the action of conjugation. Consider the 1-parameter family $f(t) = -e_1(\cos te_1 + \sin te_2) \in \mathrm{Spin}(n)$. The derivative is $\frac{d}{dt}f(t)|_{t=0} = -e_1e_2$. We have $\pi f(t)(e_j) = f(t)e_jf(t)^{-1}$, so

$$\frac{d}{dt}|_{t=0}\pi f(t)(e_j) = [\frac{d}{dt}|_{t=0}f(t), e_j] = [-e_1e_2, e_j],$$

which is 0 if $j \neq 1, 2$. If $j = 1, 2$,

$$\begin{aligned}[-e_1e_2, e_1] &= -e_1e_2e_1 + e_1e_1e_2 = -e_2 - e_2 = -2e_2, \\ [-e_1e_2, e_2] &= -e_1e_2e_2 + e_2e_1e_2 = e_1 + e_1 = 2e_1,\end{aligned}$$

so the matrix which has a 2×2-block $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ in the upper left corner and zeroes elsewhere corresponds to $-e_1e_2$, and in general one can prove that $(a_{ij}) \in \mathfrak{so}(n)$ corresponds to $-\frac{1}{4}a_{ij}e_ie_j \in \mathfrak{spin}(n)$. This formula can be used to prove the claim about $[\nabla_i, \nabla_j]$. We now need to prove that

$$\frac{-1}{8}\sum_{i\neq j} R_{ijkl}e_ie_je_ke_l = \frac{s}{4} = \sum_{i,j}\frac{R_{ijij}}{4}.$$

This can be seen using various symmetries of R together with the Clifford relations. □

Corollary 71. *If X is a compact Riemannian manifold with a spin structure and positive scalar curvature everywhere, then ind$(D) \leq 0$.*

Proof. It is enough to see that $\ker(D) = 0$, and this is true because if $D\psi = 0$, then the formula $D^2 = \nabla^*\nabla + \frac{s}{4}$ tells us that $\nabla^*\nabla\psi = -\frac{s}{4}\psi$. Integrating over X we get

$$\int_X \langle \nabla^*\nabla\psi, \psi\rangle = -\int_X \frac{s}{4}\langle \psi, \psi\rangle.$$

The right hand side is less than zero if ψ does not vanish everywhere, and the left hand side is greater than or euqal to zero. □

4.6 Spin-c structures

Define $\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} U(1)$ (that is, we take the product of the two groups and mod out by $\mathbb{Z}_2$ on both factors). For example, $\mathrm{Spin}^c(3) = SU(2) \times_{\mathbb{Z}_2} U(1) = U(2)$. Here the c stands for "complex".

Definition 72. Let X be a Riemannian, oriented n-manifold. A Spin^c structure (or spin-c structure) on X is a lift of the frame bundle to a principal $\mathrm{Spin}^c(n)$-bundle.

If V is a (complex) representation of $\mathrm{Cl}(n)$ then it determines a representation of $\mathrm{Spin}^c(n)$ by taking the representation of $\mathrm{Spin}(n)$ and tensoring it with the representation of $U(1)$, and the associated vector bundle S is still a module over $\mathrm{Cl}(TX)$, so we still have Clifford multiplication $\mathrm{cl} : TX \to \mathrm{End}(S)$ satisfying $\mathrm{cl}(a)\mathrm{cl}(b) = -\mathrm{cl}(b)\mathrm{cl}(a) = -2\langle a, b\rangle$. The point is that spin-c structures exist more often: If there is a spin structure, there is a spin-c structure, but not conversely.

Recall that X has a spin structure, if and only if $w_2(X) = 0 \in H^2(X; \mathbb{Z}_2)$.

Fact 73. *X has a spin-c structure if and only if $w_2(X)$ has an integral lift in $H^2(X; \mathbb{Z})$ – it doesn't have to be zero, but it just needs to be a mod 2 reduction of an integral class.*

This implies that the obstruction to existence of a spin-c structure is an element of $H^3(X; \mathbb{Z})$: Consider the sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0.$$

This gives rise to a Bockstein sequence

$$\cdots \to H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z}_2) \to H^3(X; \mathbb{Z}) \to \cdots .$$

Every closed oriented 3- or 4-manifold has a spin-c structure. Recall that the set of spin structures on X, if non-empty, is an affine space over $H^1(X; \mathbb{Z}_2)$.

Fact 74. *The set of spin-c structures on X, if $\neq \emptyset$, is an affine space over $H^2(X; \mathbb{Z})$.*

Another nice thing is that if you buy a spin-c structure, you get a free line bundle! We have a map $\mathrm{Spin}^c(n) \to U(1)$, given by $[g, u] \mapsto u^2$, for $g \in \mathrm{Spin}(n), u \in U(1) \subseteq \mathbb{C}$. In the case of $\mathrm{Spin}^c(3) = U(2)$, this just corresponds to the determinant. In other words, a Spin^c-bundle gives a $U(1)$-bundle, which is the same as a line bundle. We denote this $\mathcal{L}$.

Fact 75. *We have $c_1(\mathcal{L}) \equiv w_2(X)$.*

If $\mathfrak{s}$ is a spin-c structure, write $c_1(\mathfrak{s}) = c_1(\mathcal{L})$.

Fact 76. *If $\alpha \in H^2(X; \mathbb{Z})$ then $c_1(\mathfrak{s} + \alpha) = c_1(\mathfrak{s}) + 2\alpha$.*

Let $\mathfrak{s}$ be a spin-c structure on X, and let S be a spin bundle. Remember that for a spin bundle, there is a canonical connection coming from the Levi–Civita connection. For spin-c structures, this is no longer the case.

A (unitary) connection on $\mathcal{L}$ determines a connection on the Spin^c-bundle as follows. We have an exact sequence

$$0 \to \mathrm{Spin}(n) \to \mathrm{Spin}^c(n) \to U(1) \to 0$$

giving rise to an exact sequence of Lie algebras

$$0 \to \mathfrak{so}(n) \to \mathfrak{spin}^c(n) \to \mathfrak{u}(1) \to 0.$$

This sequence canonically splits, $\mathfrak{spin}^c(n) = \mathfrak{so}(n) \oplus \mathfrak{u}(1)$. We now want a 1-form on the Spin^c-bundle $F \to \mathrm{Fr} \to X$ with values in $\mathfrak{spin}^c(n)$ satisfying various properties. If we have a connection in $\mathcal{L}$, this defines an element of $\mathfrak{u}(1)$, which gives an element of $\mathfrak{spin}^c(n)$ up to an element in $\mathfrak{so}(n)$, provided by the Levi–Civita connection as before.

Let A be a connection on $\mathcal{L}$. Let ∇_A be the associated connection on the Spin^c-bundle, which we denote S now. This defines a Dirac operator $D_A : \Gamma(S) \to \Gamma(S)$ as before. We have a formula

$$D_A^2 = \nabla_A^* \nabla_A + \frac{s}{4} + \frac{1}{2}\mathrm{cl}(F_A).$$

11th lecture, October 5 2010

Let X be a closed oriented Riemannian manifold. A spin-c structure $\mathfrak{s}$ on X is a lift of $\mathrm{Fr}(X)$ to a principal $\mathrm{Spin}^c(n)$-bundle, where $\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} \times U(1)$, where $\mathbb{Z}_2$ acts by the covering transformation on the left and by negating on the right. There is a map $\mathrm{Spin}^c(n) \to U(1)$ given by $(g, z) \mapsto z^2$, which determines a complex linebundle $\det(\mathfrak{s})$. The set of Spin^c structures up to isomorphism is an affine space over $H^2(X; bbZ)$ if the set is nonempty (which is the case for dimension 3,4). Given $\mathfrak{s}$, any spin representations of $\mathrm{Cl}(n)$ gives an associated vector bundle S with Clifford multiplication $\mathrm{cl} : TX \otimes S \to S$. A connection on $\det(\mathfrak{s})$ determines a connection ∇_A on S respects cl. This gives rise to a Dirac operator D_A. Recall that the spin representations of $\mathrm{Cl}(3)$ are given by

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The spin representations of $\mathrm{Cl}(4)$ are

$$e_0 \mapsto \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad e_i \mapsto \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}.$$

In fact, the Clifford module S (the spin bundle together with the Clifford multiplication on it) is equivalent to $\mathfrak{s}$. This is the viewpoint of [KM] – this explains why spin-c structures are interesting: They are exactly the ones that give rise to the Clifford multiplication we want. Note that in 3 dimensions we get a $\mathbb{C}^2$-bundle $S \to X^3$ with a Dirac operator $D_A : \Gamma(S) \to \Gamma(S)$, and in 4 dimensions a $\mathbb{C}^2 \oplus \mathbb{C}^2$-bundle $S_+ \oplus S_- \to X^4$. If $v \in TX$, then $\mathrm{cl}(v)$ exchanges S_+ and S_-, and $D_A : \Gamma(S_\pm) \to \Gamma(S_\mp)$.

Theorem 77 (Bochner–Weitzenböck–Lichnerowicz formula)**.** *We have*

$$D_A^2 = \nabla_A^* \nabla_A + \frac{s}{4} + \frac{1}{2} cl(F_A).$$

Here $\mathrm{cl} : \bigwedge^* T^*X \to \mathrm{End}(S)$ is defined by

$$\mathrm{cl}(\alpha \wedge \beta) = \frac{1}{2}(\mathrm{cl}(\alpha)\mathrm{cl}(\beta) + (-1)^{|\alpha||\beta|}\mathrm{cl}(\beta)\mathrm{cl}(\alpha))$$

Proof of the formula. At a point, arrange it so that $\nabla_{e_i} e_j = 0$. In a local trivialization the connection on $\det(\mathfrak{s})$ has the form $\sum_i (\partial_i A_i) \otimes dx_i$, where A_i is an imaginary valued function. Now

$$\nabla_A = \sum_i (\partial_i + \frac{1}{2} A_i) \otimes dx_i.$$

Here the $\frac{1}{2}$ comes from the z^2 in the definition of the bundle; in general $\nabla_{A+a} = \nabla_A + \frac{1}{2}a$. Using the summation convention, we also have

$$\nabla_A^*(\alpha_i\, dx_i) = (-\partial_i - \frac{1}{2}A_i)\alpha_i,$$

and denoting by $\nabla_{A,i}$ the connection ∇_A in the direction i.

$$\begin{aligned} D_A^2 \psi &= e_i \nabla_{A,i}(e_j \nabla_{A,j} \psi) \\ &= e_i e_j \nabla_{A,i} \nabla_{A,j} \psi \\ &= -\nabla_{A_i} \nabla_{A,i} \psi + \frac{1}{2} \sum_{i \neq j} e_i e_j [\nabla_{A,i}, \nabla_{A,j}] \psi. \end{aligned}$$

Here the first term is the $\nabla_A^* \nabla_A \psi$. The $[\nabla_{A_i}, \nabla_{A,j}]$ is the $dx_i\, dx_j$ component of $F(\nabla_A)$. The Levi–Civita part of the connection ∇_A gives the $\frac{s}{4}$, the other part the $\frac{1}{2}\mathrm{cl}(F_A)$. Denoting the spin-c structure by F, we have a diagram as below (the horizontal arrows denoting the bundle action as is usual).

$$\begin{tikzcd}
\mathrm{Spin}^c(n) \arrow[r] & F \arrow[r] \arrow[d] & \det(\mathfrak{s}) \\
SO(n) \arrow[r] & \mathrm{Fr}(X) \arrow[d] & \\
& X &
\end{tikzcd}$$

We have $\mathfrak{spin}^c(n) = \mathfrak{so}(n) \oplus \mathfrak{u}(1)$, and $\mathfrak{so}(n) \oplus \mathfrak{u}(1) \to \mathfrak{u}(1)$ is given by $(x, y) \mapsto 2y$. That is, the curvature 2-form on F arrises as the sum of the pullbacks of curvatures on $\mathrm{Fr}(X)$ and $\det(\mathfrak{s})$., so the $F(\nabla_A)$ is the sum of the curvature of the Levi–Civita connection and $\frac{1}{2}F_A$. Write $F_A = \sum_{i<j} F_{ij} dx_i\, dx_j$, and $F_{ji} = F_{ij}$. Then we can also write $F_A = \frac{1}{2}\sum_{i\neq j} F_{ij} dx_i\, dx_j$. Then, by the definition above,

$$\mathrm{cl}(F_A) = \frac{1}{2}\sum_{i\neq j} F_{ij}\mathrm{cl}(e_i)\mathrm{cl}(e_j).$$

□

Claim 78. *In the 4-dimensional time, let $\omega \in \Omega^2(X)$. Then $cl(\omega) : S_\pm \to S_{mp}$. If $*\omega = \omega$, then $cl(\omega)S_- = 0$. If $*\omega = -\omega$ then $cl(\omega)S_+ = 0$. Note that any 2-form can be written $\omega = \omega^+ + \omega^-$, where $\omega^+ = (\omega + *\omega)/2$ and $\omega^- = (\omega - *\omega)/2$. We have $*^2 = 1$ on Ω^2, and*

$$\begin{pmatrix} cl(\omega^+) & 0 \\ 0 & cl(\omega^-) \end{pmatrix}.$$

Proof. Consider for example the calculation

$$\begin{aligned}
\mathrm{cl}(e_0e_1 + e_2e_3) &= \tfrac{1}{2}(\mathrm{cl}(e_0)\mathrm{cl}(e_1) - \mathrm{cl}(e_1)\mathrm{cl}(e_0) + \mathrm{cl}(e_2)\mathrm{cl}(e_3) - \mathrm{cl}(e_3)\mathrm{cl}(e_2)) \\
&= \tfrac{1}{2}\left(\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}\right. \\
&\quad \left. - \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\right) \\
&= \tfrac{1}{2}\left(2\begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} + 2\begin{pmatrix} -\sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}\right) = 2\begin{pmatrix} -\sigma_1 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Here we have used that $\sigma_1\sigma_2 = .\sigma_3$, $\sigma_2\sigma_3 = -\sigma_1$, $\sigma_3\sigma_1 = -\sigma_2$. □

5 4-dimensional Seiberg–Witten equations

5.1 The equations

Fix a spin-c structure. The inputs are a connection on $\det(\mathfrak{s}) = \det(S_+) = \det(S_-)$, a section ψ on S_+. The equations are

$$D_A\psi = 0,$$
$$\frac{1}{2}\mathrm{cl}(F_A^+) = (\psi\psi^*)_0,$$

where $\psi\psi^*$ is the matrix defined by taking inner products, and the $_0$ denotes trace-free part. That is, if $\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, then

$$\psi\psi^* = \begin{pmatrix} |\alpha|^2 & \alpha\overline{\beta} \\ \overline{\alpha}\beta & |\beta|^2 \end{pmatrix},$$

and

$$(\psi\psi^*)_0 = \begin{pmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|^2) & \alpha\overline{\beta} \\ \overline{\alpha}\beta & \frac{1}{2}(|\beta|^2 - |\alpha|^2) \end{pmatrix}.$$

Note that cl defines an isomorphism from self-dual 2-forms to trace-free endormorphisms of S_+, $\mathrm{cl} : \bigwedge_+^2 \to \mathrm{End}(S_+)_0$, so the last of the Seiberg–Witten equations can be thought of as an equation in $\mathrm{End}(S_+)_0$.

5.2 Gauge transformations

Consider the automorphism group of $\mathfrak{s}$, $\mathcal{G} = \mathrm{Aut}(\mathfrak{s}) = \mathrm{Maps}(X, S^1)$ (the last equality can be seen since any automorphism $g : U \to \mathrm{Spin}^c(n)$ gives rise to an automorphism g on S, which can be seen as a map $X \to S^1$, since $\mathrm{cl}(v)(g\psi) = g\mathrm{cl}(v)(\psi)$). A map $g : X \to S^1$ acts as

$$(A, \psi) \mapsto (A - 2g^{-1}dg, g\psi).$$

We see that (A, ψ) solves the Seiberg–Witten equation if and only if $g(A, \psi)$ does. Note first that curvature is invariant under automorphism; the ψ gets multiplied by g and ψ^* by $\overline{g}$, so that $(\psi\psi^*)_0$ doesn't change either. Therefore the second equation is satisfied. For the first one, note

$$\begin{aligned} D_{A-2g^{-1}dg}(g\psi) &= \mathrm{cl}(\nabla_A(g\psi) - g^{-1}dg \cdot g\psi) \\ &= \mathrm{cl}(dg \cdot \psi + g\nabla_A\psi - dg \cdot \psi) = g \cdot D_A\psi. \end{aligned}$$

Let $\mathcal{B}$ denote denote the set of solutions to the Seiberg–Witten equations modulo $\mathcal{G}$. Under favorable circumstances, this will be a finite set of points that we can count, due to marvelous analytical properties of the equations. The Seiberg–Witten equations, in an appropriate sense, are elliptic, so that $\mathcal{B}$ is a finite-dimensional manifold, generically. They also exhibit compactness, as we will discuss next time.

12th structure, October 7th 2010

Recall first the Seiberg–Witten equations: X is a closed oriented Riemannian 4-manifold with a spin-c structure $\mathfrak{s}$, giving rise to the spin bundle

$$\begin{array}{ccc} \mathbb{C}^2 \oplus \mathbb{C}^2 & \longrightarrow & S = S_+ \oplus S_- \\ & & \downarrow \\ & & X \end{array}$$

with Clifford multiplication $\mathrm{cl} : TX \otimes S_\pm \to S_\mp$. A connection A on $\det(\mathfrak{s}) = \det(S_+)$ determinse a connection ∇_A on S_{pm}. Considering only the S_+ part, we get a Dirac operator

$$D_A : \Gamma(S_+) \xrightarrow{\nabla_A} \Gamma(T^*X \otimes S_+) \xrightarrow{\mathrm{cl}} \Gamma(S_-).$$

Like the full Dirac operator, this is elliptic.

Theorem 79. *The index of the above "half" Dirac operator is*

$$ind(D_A) = \frac{c_1(\mathfrak{s})^2 - \sigma(X)}{8},$$

where $\sigma(X)$ is the signature of X, which by definition is the signature of the cup product pairing $H^2(X;\mathbb{Z}) \otimes H^2(X;\mathbb{Z}) \to H^4(X;\mathbb{Z}) = \mathbb{Z}$, also sometimes written as $\sigma(X) = b_2^+(X) - b_2^-(X)$.

This theorem is a special case of the Atiyah–Singer index theorem; see [LM].

For A a connection on $\det(\mathfrak{s})$, ψ a section of S_+ the equations are

$$D_A\psi = 0,$$
$$\frac{1}{2}\mathrm{cl}(F_A^+) = (\psi\psi^*)_0.$$

In the last equation, we compare the two expressions via the isomorphism $\mathrm{cl} : \bigwedge_+^2 \xrightarrow{\cong} \mathfrak{su}(S_+)$. Here $_0$ denotes trace-free part.

The set of gauge transformations $\mathcal{G} = \mathrm{Aut}(\mathfrak{s}) = \mathrm{Maps}(X, S^1)$ acts on the set of solutions by $g(A,\psi) = (A - 2g^{-1}dg, g\psi)$, and we are interested in the space of solutions modulo gauge transformations; denote this set by $\mathcal{B}$.

Theorem 80. *Assume X is closed. $\mathcal{B}$ is compact; if $\{(A_n,\psi_n)\}$ is a set of solutions, then after passing to a subsequence and applying suitable gauge transformations, $(A_n,\psi_n) \to (A_\infty,\psi_\infty)$ in the C^∞ topology.*

Lemma 81. *Assume X is closed. We have an a priori estimate: If (A,ψ) is any solution to the Seiberg–Witten equation, then $|\psi|^2 \leq \max_X\{0, -cs\}$, where s denotes scalar curvature, and c is a suitable constant, which we'll figure out in the proof.*

Proof. Suppose (A,ψ) is a solution to the equations, and let p be a point where $|\psi|^2$ is maximized, which is possible since X is closed. At p, $\Delta|\psi|^2 \geq 0$. Here $\Delta = d^*d$, and in local coordinates where $g_{ij} = \delta_{ij} +$ second order terms, and we have $\Delta = -\sum_i \frac{\partial^2}{\partial x_i^2}$ at p. Expanding, we get

$$\begin{aligned}
0 \leq \Delta|\psi|^2 &= -\sum_i \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\langle\psi,\psi\rangle \\
&= -\sum_i \frac{\partial}{\partial x_i}(\langle\nabla_{A,i}\psi,\psi\rangle + \langle\psi,\nabla_{A,i}\psi\rangle) \\
&= \sum_i -2\mathrm{Re}\langle\nabla_{A,i}^2\psi,\psi\rangle - 2\langle\nabla_{A,i}\psi,\nabla_{A,i}\psi\rangle \\
&= 2\mathrm{Re}\langle\nabla_A^*\nabla_A\psi,\psi\rangle - 2|\nabla_A\psi|^2 \\
&\leq 2\mathrm{Re}\langle\nabla_A^*\nabla_A\psi,\psi\rangle \\
&= \mathrm{Re}\langle 2D_A^2\psi - \frac{s}{2}\psi - \mathrm{cl}(F_A^+)\psi,\psi\rangle \\
&= -\frac{s}{2}|\psi|^2 - \mathrm{Re}\langle 2(\psi\psi^*)_0\psi,\psi\rangle.
\end{aligned}$$

Here, we used that $d\langle\alpha,\beta\rangle = \langle\nabla_A\alpha,\beta\rangle + \langle\alpha,\nabla_A\beta\rangle$ and $D_A^2 = \nabla_A^*\nabla_A + \frac{s}{4} + \frac{1}{2}\mathrm{cl}(F_A)$, and the fact that F_A^- acts as 0 on ψ.

Now, using the expression for $(\psi\psi^*)_0$ given last time,

$$\begin{aligned}
\langle(\psi\psi^*)_0\psi,\psi\rangle &= (\overline{\alpha}\ \overline{\beta})(\psi\psi^*)_0\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \\
&= (\overline{\alpha}\ \overline{\beta})\begin{pmatrix}\frac{1}{2}|\alpha^2\alpha + \frac{1}{2}|\beta|^2\alpha \\ \frac{1}{2}|\beta|^2\beta + \frac{1}{2}|\alpha|^2\beta\end{pmatrix} \\
&= \frac{1}{2}|\alpha|^4 + |\alpha|^2|\beta|^2 + \frac{1}{2}|\beta|^4 \\
&= \frac{1}{2}|\psi|^4.
\end{aligned}$$

All in all we now have $0 \leq -\frac{s}{2}|\psi|^2 - |\psi|^4$, when $|\psi|^2$ is maximized. If $\psi \neq 0$, then $|\psi|^2 \leq -\frac{s}{2}$, and c turned out to be $\frac{1}{2}$. $\square$

It is worth noting the importance on the equations in this proof. For example changing the sign in the other equation, wouldn't give the bound we need. Note also that in particular we have proved that for nontrivial solutions, the scalar curvature has to be negative at the point above.

5.3 Review of Sobolev spaces

Let E be a Hermitian vector bundle over a compact Riemannian manifold X. Choose any connection on E, and extend this to $\nabla : \Gamma(\otimes^k T^*X \otimes E) \to \Gamma(\otimes^{k+1}T^*X \otimes E)$, using the Leibniz rule. If $\psi \in \Gamma(E)$, then $\nabla^k \psi$ encodes the k'th derivative of ψ.

Let k be a nonnegative integer and $p \in [1, \infty]$. We define the *Sobolev norm*

$$\|\psi\|^2_{L^p_k} = \sum_{i=0}^{k} \|\nabla^i \psi\|^2_{L^p}.$$

Define $L^p_k(E)$ to be the completion of $C^\infty(X, E)$ with respect to this norm. Intuitively, these are sections with the first k derivatives in L^p. This is a Banach space.

Given k and p as above, define the *conformal weight* $w = k - \frac{n}{p}$ where $n = \dim X$.

Theorem 82 (Sobolev embedding theorem). *Given k, p, k', p', with w, w' as above. If $k \geq k'$, $w \geq w'$, then $\|\psi\|_{L^{p'}_{k'}} \leq c\|\psi\|_{L^p_k}$ for some constant c, when ψ is smooth, so $L^p_k(E) \subseteq L^{p'}_{k'}(E)$.*

For example, if $k - \frac{n}{p} \geq k'$, then $L^p_k(E) \subseteq L^\infty_{k'} = C^{k'}(E)$.

Theorem 83 (Rellich lemma). *If $k > k'$, and $w > w'$, then the inclusion $L^p_k(E) \subseteq L^{p'}_{k'}(E)$ is compact, meaning that a bounded sequence on the left hand side maps to a sequence with a convergent subsequence.*

Theorem 84 (Sobolev multiplication theorem). *We have a well-defined multiplication on Sobolev spaces*

$$L^{p_1}_{k_1}(E_1) \times L^{p_2}_{k_2}(E_2) \to L^p_k(E_1 \otimes E_2),$$

when $k \leq \min\{k_1, k_2\}$, and $w \leq \{w_1, w_2, w_1 + w_2\}$.

This theorem follows from the Sobolev embedding theorem (exercise).

If $D : C^\infty(E) \to C^\infty(F)$ is a differential operator of order m, then it extends to a map of Sobolev spaces $D : L^p_k(E) \to L^p_{k-m}(E)$. If D is elliptic, then $\|\psi\|_{L^p_k} \leq c(\|D\psi\|_{L^p_{k-m}} + \|\psi\|_{L^p})$. For example, if $\psi \in L^p_m$ and $D\psi = 0$, then ψ is smooth. Also, if D is elliptic, the map $D : L^p_k(E) \to L^p_{k-m}(F)$ is Fredholm, meaning that the dimensions of the kernel and cokernel are finite, and that the image is closed.

Assume that $p \geq 2$. If ψ is orthogonal to $\ker(D)$ then $\|\psi\|_{L^p_k} \leq c\|D\psi\|_{L^p_{k-m}}$. This follows from the open mapping theorem.

13th lecture, October 12 2010

Compactness of the solution space

Let X be a closed connected oriented Riemannian 4-manifold with $\mathfrak{s}$ as spin-c structure on X. The Seiberg–Witten equations are

$$D_A \psi = 0,$$
$$\frac{1}{2}\mathrm{cl}(F_A^+) = (\psi\psi^*)_0,$$

where A is a connection on $\det(\mathfrak{s})$ and ψ a section of S_+. Let $\mathcal{M}$ (previously $\mathcal{B}$) denote the set of solutions (A, ψ) modulo the set of gauge transformations $\mathcal{G} = \mathrm{Maps}(X, S^1)$ acting on solutions by $g(A, \psi) = (A - 2g^{-1}dg, g\psi)$.

Theorem 85. *$\mathcal{M}$ is compact (in the C^∞ quotient topology). I.e. if $\{(A_n, \psi_n)\}$ is a sequence of solutions, after applying gauge transformations g_i, a subsequence converges in C^∞ (meaning that all derivatives converge – that the sequence converges in C^k for all k).*

Last time we proved the following.

Lemma 86. *If (A,ψ) is a solution then $|\psi|^2 \le c\max_X\{0,-s\}$ for a suitable constant c.*

Proof of theorem. The proof goes in two steps: Gauge fixing and elliptic bootstrapping. Fix a smooth reference connection A_0 on $\det(\mathfrak{s})$. Any other connection is $A = A_0 + a$, $a \in \Omega^1(X; i\mathbb{R})$. We can rewrite the Seiberg–Witten equations as

$$D_{A_0}\psi + \frac{1}{2}a\cdot\psi = 0,$$
$$\frac{1}{2}\mathrm{cl}(F^+_{A_0} + d^+a) = (\psi\psi^*),$$

where in $a\cdot\psi$ we use Clifford multiplication, and d^+ is the composition $\Omega^1 \xrightarrow{d}\to \Omega^2_+$, where the last map is projection. We now need the following lemma.

Lemma 87. *There exists c such that for any a we can apply gauge transformation to arrange that $d^*a = 0$ and $|a_h| < c$, where a_h is the harmonic part of a with respect to the decomposition $\Omega^1 = \mathcal{H}^1 \oplus d\Omega^0 \oplus d^*\Omega^2$.*

Proof of lemma. Try $g = e^{i\theta}$, $\theta : X \to \mathbb{R}$. Maps of this form are exactly the one representing 0 in $H^1(X;\mathbb{Z})$, which we identified with $\mathrm{Maps}(X, S^1)$ earlier. We then have

$$d^*(a - 2g^{-1}dg) = d^*(a - 2id\theta) = d^*a - 2i\Delta\theta.$$

We can find θ with $d^*(a - 2g^{-1}dg) = 0$ provided that there exists θ with $\Delta\theta = -\frac{1}{2}id^*a$. This is possible since $\Delta : C^\infty(X,\mathbb{C}) \to C^\infty(X,\mathbb{C})$ is self-adjoint, $\Delta^* = \Delta$, so f is in $\mathrm{Im}(\Delta)$, if and only if $f \perp \mathrm{Ker}(\Delta)$, and $\mathrm{Ker}(\Delta)$ is the constant functions, so f is in $\mathrm{Im}(\Delta)$ if and only if $\int_X f d\mathrm{vol} = 0$. We thus want $0 = \int d^*a \cdot d\mathrm{vol}$. We have

$$\int_X d^*a\cdot d\mathrm{vol} = \int_X -d(*a) = 0,$$

where the first equality can be seen by checking it at a point. We thus have the first equation, $d^*a = 0$. To get the second one, we need to use a different homotopy class than 0. To change a_h (without disturbing $d^*a = 0$), take g to be a harmonic representative of a nonzero element of $H^1(X;\mathbb{Z})$. I.e. let $\alpha \in \Omega^1(X;\mathbb{R})$ be a harmonic 1-form representing a class in $\mathrm{im}(H^1(X;\mathbb{Z}) \to H^1(X;\mathbb{R}))$. We can write $2\pi i\alpha = dg$, where $g : X \to S^1$ – alternatively, $g = e^{2\pi i\theta}$, where $\alpha = d\theta$, with $\theta : X \to \mathbb{R}/\mathbb{Z}$. Tbe following lemma is an exercise.

Lemma 88. *If α is harmonic, then the harmonic part of $g^{-1}dg$ is a constant times α and $d^*(g^{-1}dg) = 0$.*

In fact, $H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$ is isomorphic to $(S^1)^{b_1(X)}$, which is compact, which gives the second claim of the lemma. □

To do the elliptic bootstrapping, we need a small modification to the Sobolev theory from last time. If $E \to X$, with X compact, let $L^p_k(E)$ be the completion of $C^\infty(X;E)$ with respect to $\|\cdot\|L^p_k$. This is the standard definition for $p < \infty$, but not for $p = \infty$. To use the theorems from last time, we need $p < \infty$. If $k - \frac{n}{p} > k'$, then there is a compact embedding $L^p_k(E) \to C^{k'}(E)$. To complete the proof of compactness, we need the following: Given a solution (a,ψ) with $d^*a = 0$ and $|a_h| < c$, there are upper bounds on $\|a\|_{L^p_k}\|, \|\psi\|_{L^p_k}\|$, where arbitrarily large weights $k - \frac{n}{p}$.

Lemma 89. *The sequence $\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^2_+$ is elliptic (see below) and its cohomology is $\mathcal{H}^0, \mathcal{H}^1, \mathcal{H}^2_+$.*

By definition, a chain complex of differential operators

$$\Gamma(X, E_0) \overset{D_0}{\to} \Gamma(X, E_1) \overset{D_1}{\to} \cdots \overset{D_{k-1}}{\to} \Gamma(X, E_k)$$

is *elliptic* if for all non-zero $\xi \in T^*_x X$, the sequence

$$(E_0)_x \overset{\sigma(D_0)\xi}{\to} (E_1)_x \to \cdots \overset{\sigma(D_{k-1})(\xi)}{\to} (E_k)_x.$$

For an elliptic complex, if we let $D = D_0 + \cdots + D_k$, then $D + D^* : \bigoplus \Gamma(X, E_{\text{even}}) \to \bigoplus \Gamma(X, E_{\text{odd}})$ (and the other way around) is elliptic and $\ker(D + D^*)$ is isomorphic to cohomology. The lemma as well as these last statemets are left as exercises. We rewrite the second of the Seiberg–Witten equations as

$$\frac{1}{2}(F^+_{A_0} + d^+ a)\sigma(\psi, \psi),$$

where $\sigma : S_+ \otimes S_+ \to i \bigwedge^2_+$ is given by $\sigma(\alpha, \beta) = \mathrm{cl}^{-1}((\alpha\beta^*)_0)$. We have the following elliptic estimate for D_{A_0}:

$$\|\psi\|_{L^p_k} \le c(\|a\psi\|_{L^p_{k-1}} + \|\psi\|_{L^p}).$$

We also have an elliptic estimate for $d^* + d^+$, using the equality $d^+ a + \sigma(\psi, \psi) - \frac{1}{2}F^+_{A_0}$ together with the triangle inequality (and using that the above lemma gives us $\|a\|_{L^p_k} < c$ for all p, k):

$$\begin{aligned} \|a\|_{L^p_k} &\le c + \|a - a_h\|_{L^p_k} \le c + C\|(d^* + d^+)(a - a_h)\|_{L^p_{k-1}} \\ &= c + C\|d^+ a\|_{L^p_{k-1}}, \end{aligned}$$

so $\|a\|_{L^p_k} \le c + C\|\sigma(\psi, \psi)\|_{L^p_{k-1}}$, for some constants.

We can now start the bootstrapping: Assume that $p > 4$. We have a bound on $\|\psi\|_{L^p}$ and by the Sobolev multiplication theorem, we get a bound on $\|\sigma(\psi, \psi)\|_{L^{p/2}}$. Using the last elliptic estimate, we get a bound on $\|a\|_{L^{p/2}_1}$. This gives a bound, by Sobolev multiplication $\|a\psi\|_{L^{p/4}}$, which by the first estimate gives a bound on $\|\psi\|_{L^{p/4}_1}$. For $k \ge 2$, we have

$$\begin{aligned} \|\psi\|_{L^{p/4}_k} &\le f(\|a\|_{L^{p/4}_{k-1}}, \|\psi\|_{L^{p/4}_{k-1}}), \\ \|a\|_{L^{p/4}_k} &\le f(\|\psi\|_{L^{p/4}_{k-1}}), \end{aligned}$$

for suitable functions (the factor of $\frac{1}{2}$ is only relevant for the first terms). We use these estimates to repeatedly increase k. So, for each k, there exists c such that $\|\psi\|_{L^{p/4}_k}, \|a\|_{L^{p/4}_k} \le c$, and by the Sobolev embedding theorem, this gives the convergence in C^k. □

14th lecture, October 14th 2010

Let X be a closed oriented Riemannian 4-manifold with $\mathfrak{s}$ a spin-c structure. The Seiberg–Witten equations are

$$\begin{aligned} D_A\psi &= 0 \\ \frac{1}{2}F^+_A &= \sigma(\psi, \psi), \end{aligned}$$

where A is a connection on $\det(\mathfrak{s})$, and ψ is a section of S_+, and $\sigma(\alpha, \beta) = \mathrm{cl}^{-1}((\alpha\beta^*)_0)$, with the map $\mathrm{cl} : i \bigwedge^2_+ \to i\mathfrak{su}(S_+)$. The action of gauge transformations is given by $g(A, \psi) = (A - 2g^{-1}dg, A\psi)$, for $g : X \to S^1$. Last time we saw that the moduli space of solutions modulo gauge transformation, $\mathcal{M}$, is compact.

5.4 Smoothness of the moduli space

We will see when $\mathcal{M}$ has the structure of a smooth manifold, which isn't always the case. Denote by $\mathcal{B}$ the space of arbitrary pairs $\{(A, \psi)\}$ modulo gauge transformations (not necessarily solutions). We can think of it by considering a fiber bundle.

$$\begin{tikzcd} \Gamma(S_) \times i\Omega^2_+ \arrow[r] & \mathcal{E} \arrow[d, bend left] \\ & \mathcal{B} \arrow[u, bend left, "\mathrm{SW}"] \end{tikzcd}$$

Solutions of the Seiberg–Witten equations can be thought of as sections of this bundle, and $\mathcal{M} = \mathrm{SW}^{-1}(0)$, so for $\mathcal{M}$ to be a smooth manifold we would like the section to be transverse to the zero section in $\mathcal{E}$. To obtain this, we perturb the second Seiberg–Witten equation, considering instead the equations

$$\begin{aligned} D_A\psi &= 0 \\ \frac{1}{2}F_A^+ &= \sigma(\psi, \psi) + i\mu, \end{aligned}$$

for some $\mu \in \Omega^2_+$, and consider instead solutions of these equations. The solution space modulo gauge transformation is still denoted $\mathcal{M}$.

Exercise 90. For any μ, $\mathcal{M}$ is still compact.

The proof of this exercise is almost the same as the previous proof, with the a priori bound gaining a factor of $|\mu|$ or something like that.

Theorem 91.
- *If $B_2^+(X) \geq 1$, and if μ is generic, then $\mathcal{M}$ is a smooth manifold of dimension*

$$\frac{2\chi(X) + 3\sigma(X) - c_1(\mathfrak{s})^2}{4}.$$

- *If $b_2^+(X) \geq 2$ then this manifold, up to cobordism, does not depend on μ or the Riemannian metric g on X.*
- *An orientation of $H^0(X) \oplus H^1(X) \oplus H^2_+(X)$ determines an orientation of $\mathcal{M}$ compatible with the above cobordism.*

We start off by trying to understand the precense of the above $b_2^+(X)$. Consider the space "configuration space" $\mathcal{C}$ of all pairs $\{A, \psi)\}$, so $\mathcal{B} = \mathcal{C}/\mathcal{G}$. Here $\mathcal{G}$ does not quite act freely on $\mathcal{C}$, and $\mathcal{B}$ is not a manifold. Consider instead $\mathcal{C}^* = \{(A, \psi) \mid \psi \not\equiv 0\}$. Now $\mathcal{G}$ acts freely on this *, for if $(A, \psi) = (A - 2g^{-1}dg, g\psi)$, then g is constant, so if ψ is not zero everywhere, then $g \equiv 1$. On the other hand, the action of $\mathcal{G}$ on $\mathcal{C} - \mathcal{C}^*$ has S^1 stabilizer.

Definition 92. The pair (A, ψ) is called *reducible* if $\psi \equiv 0$. Otherwise *irreducible.*

Note that the word "reducible" not really makes sense in this context but comes from the notion of a reducible connection in Yang–Mills theory, where similar problems as the above arise.

Define $\mathcal{B}^* =^* \mathcal{C}/\mathcal{G}$ and let $\mathcal{M}^* = \mathcal{M} \cap \mathcal{B}^*$ be the set of irreducible solutions (A, ψ) modulo gauge transformation. We can't really hope for $\mathcal{M}$ to be a manifold in general, and only $\mathcal{M}^*$ will be.

Proof of the first part of the theorem. The proof consists of two parts

(i) If μ is generic, then $\mathcal{M}^*$ is a smooth manifold of the claimed dimension.

(ii) If $b_2^+ \geq 1$, and μ is generic, then $\mathcal{M} = \mathcal{M}^*$.

Lemma 93. *The space of all self-dual 2-forms can be decomposed $\Omega^2_+ = \mathcal{H}^2_+ \oplus d^+\Omega^1$, where d^+ is the composition $\Omega \to \Omega^2 \to \Omega^2_+$.*

Proof. Recall the Hodge decomposition $\Omega^2 = \mathcal{H} \oplus d\Omega^1 \oplus d^*\Omega^3$. Let $\omega \in \Omega^2_+$. Write $\omega = h + d\alpha + d^*\beta$ with respect to the Hodge decomposition. Remember that $d^* = \pm * d*$, so $*h$ is again harmonic. Now

$$*\omega = *h + *d\alpha + *d^*\beta = *h \pm d^*(*\alpha) \pm d * \beta.$$

By uniqueness of the composition, $h = *h$, $d\alpha = \pm d(*\beta)$ and $d^*\beta = \pm d^*(*\alpha) = *d\alpha$. We have $d\alpha + d^*\beta = d\alpha + *d\alpha$, so $d\alpha + d^*\beta \in d^+\Omega_1$. $\square$

Now fix μ. When do reducible solutions exist? The equation $D_A\psi = 0$ automatically holds if $\psi \equiv 0$. The equation $\frac{1}{2}F_A^+ = \sigma(\psi, \psi) + i\mu$ becomes $\frac{1}{2}F_A^+ = i\mu$. If we write $A = A_0 + a$, $a \in i\Omega^1$, then the equation becomes $\frac{1}{2}(F_{A_0}^+ + d^+a) = i\mu$ or

$$\frac{1}{2}d^+a = i\mu - \frac{1}{2}F_{A_0}^+.$$

So a reducible solution exists if and only if $i\mu - \frac{1}{2}F_{A_0}^+ \in id^+\Omega^+$, or $\mu + \frac{i}{2}F_{A_0}^+ \in d^+\Omega^1$, which is true if and only if the harmonic part of $\mu + \frac{i}{2}F_{A_0}^+$ is 0. It is an element of $\mathcal{H}_2^+$, so we have a codimension $b_2^+(X)$ condition on μ.

Let us prove that $\mathcal{M}^*$ is smooth for generic μ. We will show that the "universal moduli space" of tuples $(mu, A, \psi) \in \Omega^2_+ \times \mathcal{C}^*$ where (A, ψ) solves the equations perturbed by μ is smooth: It is a fiber bundle over Ω^2_+ with fibers $\tilde{\mathcal{M}}^*$. We can then apply the Sard–Smale theorem to conclude that generic μ is a regular value of the projection to Ω^2_+. We have to use appropriate Sobolev space completions so that everything is a Banach manifold. We will ignore this issue and assume that everything is okay with C^∞ functions.

Write $A = A_0 + a$. Then

$$(\mu, a, \psi) \mapsto (D_{A_0}\psi + \frac{1}{2}a\psi, \frac{1}{2}F_{A_0}^+ + \frac{1}{2}d^+a - \sigma(\sigma, \sigma) - i\mu)$$

is a map $\Omega^2_+ \times \mathcal{C}^* \to \Gamma(S_-) \times i\Omega^2_+$. We will show that the derivative of this map is surjective and use the inverse function theorem (in a Banach manifold) to conclude that the zero set is smooth. The derivative is

$$(\dot{\mu}, \dot{a}, \dot{\psi}) \mapsto (D_{A_0}\dot{\psi} + \frac{1}{2}\dot{a}\psi, \frac{1}{2}d^+\dot{a} - \sigma(\dot{\psi}, \psi) - \sigma(\psi, \dot{\psi}) - i\dot{\mu}).$$

Suppose that $(x, y) \in \Gamma(S_-) \times i\Omega^2_+$ which is orthogonal to the image, and let us show that $x = y = 0$. Taking $\dot{a} = 0, \dot{\psi} = 0$, then $y = 0$ (and here we use the pertubation term). We know that x is perpendicular to the image of $((\dot{a}, \dot{\psi}) \mapsto D_A\dot{\psi} + \frac{1}{2}\dot{a}\psi)$. That is,

$$\langle x, D_A\dot{\psi} + \frac{1}{2}\dot{a}\psi\rangle = 0.$$

Since the Dirac operator is self-adjoint, this becomes

$$\langle D_Ax, \dot{\psi}\rangle + \langle x, \frac{1}{2}\dot{a}\psi\rangle = 0.$$

Taking $\dot{a} = 0$ then x satisfies the Dirac equation $D_Ax = 0$. Since ψ by assumption is nonzero somewhere, we get that x vanishes on an open set. Now there is a unique continuation property for the Dirac equation saying that $x = 0$ – this is proved in §7 of [KM]. $\square$

15th lecture, October 19 2010

5.5 Dimension of the moduli space

Let X be a closed oriented connected Riemannian manifold, $\mathfrak{s}$ a spin-c structure on X. The Seiberg–Witten equations (with perturbations are)

$$\begin{aligned} D_A\psi &= 0 \\ \frac{1}{2}F_A^+ &= \sigma(\psi,\psi) + i\mu, \end{aligned}$$

where A is a connection on $\det(\mathfrak{s})$ and $\psi \in \Gamma(S_+)$.

If $b_2^+(X) > 0$, then for generic μ, $\mathcal{M}(X,\mathfrak{s})$ is a smooth manifold. What is its dimension?

Remember that $\mathcal{M}(X,\mathfrak{s}) = \{\text{solutions}\}/\mathcal{G}$, where $\mathcal{G} = \mathrm{Maps}(X, S^1)$ acts as $g(A,\psi) = (A - 2g^{-1}dg, g\psi)$. We fix a reference ("fiducial" in Taubes' papers, a word not appearing anywhere else) connection A_0 and write $A = A_0 + a$ for $a \in \Omega^1(X; i\mathbb{R})$. We rewrite the equations as

$$\begin{aligned} D_{A_0}\psi + \frac{1}{2}a\psi &= 0 \\ \frac{1}{2}F_{A_0}^+ + \frac{1}{2}d^+a - \sigma(\psi,\psi) - i\mu &= 0 \end{aligned}$$

We can consider the equations as a map SW. We have maps

$$i\Omega^0(X) \to i\Omega^1(X) \times \Gamma(S_+) \overset{D_{(A,\psi)}\mathrm{SW}}{\to} i\Omega^2_+(X) \times \Gamma(S_-)$$

where the first map is the derivative of the gauge group action, so it maps $\dot{g} \mapsto (-2d\dot{g}, \dot{g}\psi)$. The second map maps

$$(\dot{a},\dot{\psi}) \mapsto (\frac{1}{2}d^+\dot{a} - \sigma(\dot{\psi},\psi) - \sigma(\psi,\dot{\psi}), \frac{1}{2}\dot{a}\psi + D_A\dot{\psi}).$$

This is a complex called the *deformation complex.* We look at its cohomology.

- $H^0 = 0$, if (A,ψ) is irreducible, i.e. $\psi \not\equiv 0$. This is true for generic μ; assume this.
- $H^2 = 0$, when the derivative of the map from Seiberg–Witten equations is surjective. This is true for generic μ as we saw last time; assume this.
- $H^1 = T\mathcal{M}(X,\mathfrak{s})$.

The dimension of $\mathcal{M}(X,\mathfrak{s})$ is then $\dim(H^1)$. The complex is elliptic: Modulo lower order terms the first map maps $\dot{g} \mapsto (-2d\dot{g}, 0)$; the first part is first order, and the other zeroth order. The second map maps $(\dot{a},\dot{\psi}) \mapsto (\frac{1}{2}d^+\dot{a}, D_A\dot{\psi})$. We see it as a complex

$$(i\Omega^0 \overset{d}{\to} i\Omega^1 \overset{d^+}{\to} i\Omega^2_+) \oplus (\Gamma(S_+) \overset{D_A}{\to} \Gamma(S_-)),$$

and this is elliptic. Then the dimension of H^1 is minus the index of the complex. This splits, and we have

$$\begin{aligned} -\dim H^1 &= \mathrm{ind}_{\mathbb{R}}(\Omega^0 \overset{d}{\to} \Omega^1 \overset{d^+}{\to} \Omega^2_+) - 2\mathrm{ind}_{\mathbb{C}}(D_A) \\ &= 1 - b_1(X) + b_2^+(X) - 2\frac{c_1(s)^2 - \sigma(X)}{8}. \end{aligned}$$

(There might be some sign stuff going on here, cf. next lecture.) We write the Euler characteristic as $\chi(X) = 2 - 2b_1(X) + b_2^+(X) + b_2^-(X)$ and the signature as $\sigma(X) = b_2^+(X) - b_2^-(X)$, so

$$\frac{\chi(X) + \sigma(X)}{2} = 1 - b_1(X) + b_2^+(X),$$

so

$$\dim(\mathcal{M}(X,\mathfrak{s})) = \frac{2\chi(X) + 3\sigma(X) - c_1(\mathfrak{s})^2}{4}.$$

5.6 Orientation of the moduli space

The moduli space $\mathcal{M}(X,\mathfrak{s})$ is orientable. An orientation of $H^0(X)\oplus H^1(X)\oplus H^2_+(X)$ (sometimes called a *homology orientation*) determines an orientation of $\mathcal{M}(X,\mathfrak{s})$.

Here's the rough idea: Any real elliptic complex $\Gamma(E_0)\overset{D_0}{\to}\Gamma(E_1)\to\cdots\overset{D_{k-1}}{\to}\Gamma(E_k)$ has a determinant

$$\det=\bigotimes_{i\text{even}}\Lambda^{\max}(H^i)\otimes\bigotimes_{i\text{odd}}\Lambda^{\max}(H^i)^*\cong\mathbb{R}.$$

If you derform the elliptic complex through elliptic complexes, this line varies smoothly: The determinant depends continuously on the elliptic complex. This would be obvious, if the dimensions of H^i didn't change, but for a general complex they can jump.

Orienting H^1 will be the same as orienting the determinant line of the deformation complex. Believing that the determinant varies continuously, the determinant line of the deformation complex, and it is enough to give an orientation of the determinant of the complex

$$(\Omega^0\overset{d}{\to}\Omega^1\overset{d^+}{\to}\Omega^2_+)\oplus(\Gamma(S_+)\overset{D_A}{\to}\Gamma(S_-)).$$

which is the tensor product of the determinants of the two complexes. The first one gets the homology orientation, and the other one is complex linear.

6 The Seiberg–Witten invariant

Assume that $b^+(X)\geq 2$, and fix a homology orientation of X. We define the *Seiberg–Witten invariant* $\mathrm{SW}(X,\mathfrak{s})\in\mathbb{Z}$ as follows: Choose a generic μ such that the moduli spaces are manifolds.

- If $2\chi+3\sigma-c_1(\mathfrak{s})^2<0$, we put $\mathrm{SW}(X,\mathfrak{s})=0$.
- If $2\chi+3\sigma-c_1(\mathfrak{s})^2=0$, the moduli space is a finite set of points, each of which has a sign associated to it, and we let $\mathrm{SW}(X,\mathfrak{s})$ be the signed count of points.
- In the case of positive dimension, there are ways to get invariants by integrating certain forms, but they always turn out to be 0, and it is not clear if you can get anything non-trivial. We do the following in the even-dimensional case:

  We have $\mathcal{M}(X,\mathfrak{s})\subseteq\mathcal{B}(X,\mathfrak{s})=\mathcal{C}(X,\mathfrak{s})/\mathcal{G}$. There is a natural S^1-bundle over this.

$$\begin{tikzcd} S^1 \arrow[r] & \tilde{B}(X,\mathfrak{s})=\mathcal{C}(X,\mathfrak{s})/\mathcal{G}_0 \arrow[d] \\ & \mathcal{B}(X,\mathfrak{s}) \end{tikzcd}$$

  where $\mathcal{G}_0=\mathrm{Maps}((X,x_0),(S^1,1))$. This pulls back to a bundle over $\mathcal{M}(X,\mathfrak{s})$. We can talk about $c_1(\mathcal{B})\in H^2(X;\mathbb{Z})$. When $2\chi+4\sigma-c_1^2>0$, define $\mathrm{SW}(X,\mathfrak{s})$ by integrating a power of $c_1(\tilde{\mathcal{B}})$ over $\mathcal{M}(X,\mathfrak{s})$. The *simple type conjecture* says that this is always zero.

  Note that if $b^+(X)\geq 2$ is actually an invariant by the result of last lecture.

6.1 Basic properties of the invariant

At this points it is not clear, that the invariant is ever non-zero. We have the following properties

- $\mathrm{SW}(X,\mathfrak{s})\neq 0$ only if $1-b_1(X)+b_2^+(X)$ is even, as is seen from the formula for the index before.
- Given X, there are only finitely many $\mathfrak{s}$ with $\mathrm{SW}(X,\mathfrak{s})\neq 0$. In fact, given μ, there are only finitely many $\mathfrak{s}$ with $\mathcal{M}(X,\mathfrak{s})\neq\emptyset$.

Proof of the last claim. Suppose $\mathcal{M}(X,\mathfrak{s}) \neq \emptyset$. Then $2\chi + 3\sigma - c_1(\mathfrak{s})^2 \geq 0$. Remember that the cohomology class of the curvature is $[F_A] = -2\pi i c_1(\mathfrak{s})$. That means that $c_1(\mathfrak{s})^2 = -\frac{1}{4\pi^2} F_A \wedge F_A]$. If α and β are 2-forms, then

$$\alpha \wedge \beta = \langle \alpha, *\beta \rangle d\mathrm{vol},$$

and

$$F_A \wedge F_A = -\langle F_A, *F_A \rangle d\mathrm{vol}.$$

Writing $F_A = F_A^+ + F_A^-$, $*F_A = F_A^+ - F_A^-$, we get that

$$\int_X F_A \wedge F_A = \int_X \left(+\|F_A^+\|^2 - \|F_A^-\|^2\right) d\mathrm{vol}.$$

The left hand side is equal to $-4\pi^2 c_1(\mathfrak{s})^2$, which is greater than or equal to some constant from before. Thus

$$\int_X (-\|F_A^+\|^2 + \|F_A^-\|^2) d\mathrm{vol}$$

is less than or equal to this constant. So far we only used Chern–Weil theory and the formula $\dim \mathcal{M}(X,\mathfrak{s}) \geq 0$, and we are yet to use the Seiberg–Witten equations. Remember the a priori estimate $|\psi|^2 \leq c$ for some constant c. The Seiberg–Witten equations say that $F_A^+ = 2\sigma(\psi,\psi) + 2i\mu$, so we have a pointwise bound $|F_A| \leq c$, so $\int \|F_A^+\|^2 d\mathrm{vol} \leq c$. By the previous estimate,

$$\int \|F_A\|^2 d\mathrm{vol} \leq c.$$

Now $\int \|F_A\|^2 d\mathrm{vol} \geq \int \|(-2\pi i c_1(\mathfrak{s}))_h\|^2 d\mathrm{vol}$, since harmonic forms minimize the L^2-norm in their equivalence class, by Hodge theory. There are only finitely many possibilities for the image of $c_1(\mathfrak{s})$ in $H^2(X;\mathbb{Z})/\mathrm{Torsion}$ (the torsion subgroup is finite, since X is a manifold). This means that there are only finitely many possibilities for $c_1(\mathfrak{s})$. This further implies that there are only finitely many possibilities for $\mathfrak{s}$. □

We will see a better version of the above, when we get to the adjunction inequality: Suppose $\Sigma \hookrightarrow X$ is a closed, oriented, connected smoothly embedded surface of genus $g > 0$. Suppose $\mathrm{SW}(X,\mathfrak{s}) \neq 0$ and that the self-intersection number $\Sigma \cdot \Sigma \geq 0$. Then

$$2g - 2 \geq |\langle c_1(\mathfrak{s}), [\Sigma] \rangle| + \Sigma \cdot \Sigma.$$

This will imply the Thom conjecture.

16th lecture, October 21st 2010

Let X be a closed oriented connected Riemannian 4-manifold with $b_2^+(X) \geq 2$,and $\mathfrak{s}$ a spin c-structure. Choose a homology orientation of $H^0 \oplus H^1 \oplus H^2_+$. Then $\mathrm{SW}(X,\mathfrak{s}) \in \mathbb{Z}$ counts solutions to the equation

$$\begin{aligned} D_A\psi &= 0 \\ \frac{1}{2} F_A^+ &= \sigma(\psi,\psi) + i\mu, \end{aligned}$$

where μ is a generic self-dual 2-form. The invariant is defined when $1 - b_1 + b_2^+ + \mathrm{ind}(D_A) = -\frac{2\chi + 3\sigma - c_1(\mathfrak{s})^2}{4} = 0$ (up to maybe some sign confusion. We saw last time how to deal with the positive case).

We continue our discussion of properties of the invariant.

- As we saw last time, there are only finitely many $\mathfrak{s}$ with $\mathrm{SW}(X, \mathfrak{s}) \neq 0$.

- If X has a metric of positive scalar curvature then $\mathrm{SW}(X, \mathfrak{s}) = 0$. This follows, as the above, from the a priori estimate; recall that

$$|\psi|^2 \leq \max_X \{0, -\mathrm{scalar}\},$$

  meaning that ψ is reducible, but we chose μ such that this wasn't the case.

- The adjunction inequality (proved by Kronheimer and Mrowka). If $\Sigma \hookrightarrow X$, $g > 0$, $\Sigma \cdot \Sigma \geq 0$. If $\mathrm{SW}(X, \mathfrak{s}) \neq 0$, then $2g - 2 \geq \Sigma \cdot \Sigma + |c_1(\mathfrak{s}) \cdot \Sigma|$. This leads to a proof (by Kronheimer and Mrowka, 1994) of the Thom conjecture: If $\Sigma \hookrightarrow \mathbb{C}P^2$ of degree d, then $2g-2 \geq (d-1)(d-2)/2$ meaning that holomorphic curves minimize genus in their homology class. The generalized Thom conjecture (proved by Ozsvath and Szabo) says that holomorphic curves are genus minimizing in any closed symplectic 4-manifold.

- If $X = X_1 \# X_2$ with $b_2^+(X_1), b_2^+(X_2) \geq 1$ then $\mathrm{SW}(X, \mathfrak{s}) = 0$. Remember that the connected sum is formed by removing a 4-ball from each manifold, and gluing a cylinder between them. By Mayer–Vietoris $H_2(X_1 \# X_2) = H_2(X_1) \oplus H_2(X_2)$, the intersection form for $X_1 \# X_2$ is the block diagonal of the individual intersection forms, so $b_2^+(X_1 \# X_2) = b_2^+(X_1) + b_2^+(X_2)$. Mayer–Vietoris also gives $H_1(X_1 \# X_2) = H_1(X_1) \oplus H_1(X_2)$. Furthermore $\mathrm{ind}(D_A^{X_1 \# X_2}) = \mathrm{ind}(D_A^{X_1}) + \mathrm{ind}(D_A^{X_2})$. Also, $\mathrm{Spin}^c(X_1 \# X_2) = \mathrm{Spin}^c(X_1) \times \mathrm{Spin}^c(X_2)$. Putting all of this together, we get

$$\dim(\mathcal{M}_{X_1 \# X_2}, \mathfrak{s}) = (\dim(\mathcal{M}_{X_1}, \mathfrak{s}|_{X_1}) + \dim(\mathcal{M}_{X_2}, \mathfrak{s}_{X_2})) + 1,$$

  the 1 coming from the fact, that H_0 is not additive with respect to connected sum. If $\dim(\mathcal{M}_{X_1 \# X_2}, \mathfrak{s}) = 0$, then $\dim(\mathcal{M}_{X_i}, \mathfrak{s}|_{X_i}) < 0$ for some i.

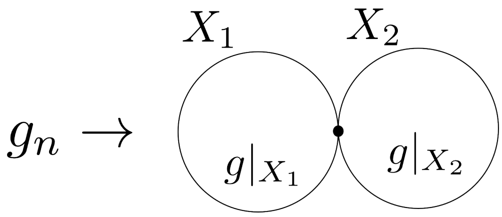

Figure 14: The metrics are pinched in a way as to make the connected sum look like something like a wedge of the spaces.

  There is a notion of pinching the metric; there is a sequence of metrics g_n on $X_1 \# X_2$, such that we have something like on Fig. 14. Also $|\mathrm{scalar}(g_n)| < C$. Suppose that $\mathrm{SW}(X, \mathfrak{s}) \neq 0$. Then for all n there exists a solution to the Seiberg–Witten equations for g_n. With an n-independent bound on $|\psi_n|$, we get solutions for $(X_1, \mathfrak{s}|_{X_1})$ and $(X_2, \mathfrak{s}|_{X_2})$. With perturbations μ_n on $X_1 \# X_2$, we have "$\lim_{n \to \infty} = \mu|_{X_1} \# \mu|_{X_2}$". Since $b_2^+(X_1), b_2^+(X_2) \geq 1$, if $\mu|_{X_1}$, $\mu|_{X_2}$ are generic, then $\mathcal{M}(X_i, \mathfrak{s}|_{X_i}) = \emptyset$.

- "Charge conjugation invariance": Every spin-c structure $\mathfrak{s}$ has a conjugate spin-c structure $\bar{\mathfrak{s}}$. For the spinor bundle use $\overline{S_+}$ instead of S_+ (meaning that multiplication by i is changed to multiplication by $-i$), and $\overline{S_-}$ instead of $\overline{S_-}$. The Clifford action is still complex linear. We have $\det(\bar{\mathfrak{s}}) = \overline{\det(\mathfrak{s})}$, so $c_1(\bar{\mathfrak{s}}) = -c_1(\mathfrak{s})$. The Seiberg–Witten equations are the same and so are the moduli spaces. The Seiberg–Witten invariant becomes $\mathrm{SW}(X, \bar{\mathfrak{s}}) = (-1)^{(\chi+\sigma)/4} \mathrm{SW}(X, \mathfrak{s})$.

We may be wondering if the invariants are ever non-zero. We will formulate a theorem by Taubes showing among other things that this is the case.

6.2 Taubes' theorem

Let (X, ω) be a symplectic 4-manifold with $b_2^+(X) \geq 2$ with symplectic form ω: a closed 2-form on X with $\omega \wedge \omega > 0$. Darboux's theorem says that locally $\omega = dx_1dx_2 + dx_3dx_4$.

Choose an ω-compatible almost complex structure $J : TX \to TX$. That is, $J^2 = -1$ and $g(v, w) = \omega(v, Jw)$ is a Riemannian metric.

Lemma 94. *The space of such J is nonempty and contractible.*

Lemma 95. *For g as above, the symplectic form ω is self-dual, and $|\omega| = \sqrt{2}$.*

At a point, $g_{ij} = \delta_{ij}$, $\omega = dx_0dx_1 + dx_2dx_3$, and $J(\partial/\partial x_0) = \partial/\partial x_1$, $J(\partial/\partial x_2) = \partial/\partial x_3$.

Suppose $\mathfrak{s}$ is a spin-c structure. Then $\mathrm{cl}(\omega) : S_+ \to S_+$ has eigenvalues $\pm 2i$: At a point as above, we can trivialize S_+, S_- such that

$$\mathrm{cl}(dx_0) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad , \mathrm{cl}(dx_i) = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix},$$

where the σ_i are as previously. Then

$$\begin{aligned}
\mathrm{cl}(\omega) &= \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \\
&= -2 \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix}, \\
\mathrm{cl}(\omega)|_{S_+} &= \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}.
\end{aligned}$$

Because this is true for every point, the $-2i$ eigenspace gives a complex line bundle E on X. The $+2i$ eigenspace gives another one, $K^{-1}E$, where K is the canonical bundle $\Lambda^{2,0}$, and $K^{-1} = \Lambda^{0,2}$. We have a map from $\mathrm{Spin}^c(X)$ to the set of line bundles on X, which is equal to $H^2(X; \mathbb{Z})$ (by an identification given by and depending on ω), given by mapping $\mathfrak{s}$ to the E above. This map is $H^2(X; \mathbb{Z})$-equivariant, so it's actually a bijection: If $\alpha \in H^2(X; \mathbb{Z})$, $c_1(\mathcal{L}) = \alpha$, then S_+ turns into $S_- \otimes \mathcal{L}$, $E \mapsto E \otimes \mathcal{L}$, and $c_1(E \otimes \mathcal{L}) = c_1(E) + c_1(\mathcal{L})$.

Use the above bijection to identify $\mathrm{Spin}^c(X) \cong H^2(X; \mathbb{Z})$.

Theorem 96 (Taubes, 1994)**.** *We have the following:*

- *$SW(X, 0) = \pm 1$ and by charge conjugation $SW(X; K) = \pm 1$.*
- *If $SW(X, e) \neq 0$, then $\langle e, [\omega] \rangle = 0$.*
- *If $SW(X, e) \neq 0$ and $\langle e, [\omega] \rangle = 0$, then $e = 0$.*

Combining this theorem with the adjunction inequality, we get the following: If Σ is a connected closed oriented surface of genus $g > 0$ in (X, ω) a symplectic manifold with $b_2^+(X) \geq 2$, and $\Sigma \cdot \Sigma = 0$, then

$$2g - 2 \geq \Sigma \cdot \Sigma + |\langle K, [\Sigma] \rangle|.$$

If Σ is holomorphic then $2g - 2 = \Sigma \cdot \Sigma + \langle K, [\Sigma] \rangle$. This is not enough to prove the Thom conjecture which is about $b_2^+(X) \geq 2$, but $\mathbb{C}P^2$ has $b_2^+(\mathbb{C}^2) = 1$.

17th lecture, October 26th 2010

As usual, let X be a closed oriented Riemannian manifold, $\mathfrak{s}$ a spin-c structure, and we have the Seiberg–Witten equations

$$\begin{aligned}
\frac{1}{2} F_a^+ &= \sigma(\psi, \psi) + i\mu \\
D_A \psi &= 0,
\end{aligned}$$

where A is a connection on $\det(\mathfrak{s})$, $\psi \in \Gamma(S_+)$, and $\mu \in \Omega^2_+$ is generic. Today we will look at the proof of Taubes' theorem, but Taubes doesn't use the $\frac{1}{2}$ in the first equation, and we won't either – this doesn't matter, as we can just change ψ and μ accordingly. Suppose X has a symplectic form ω. We choose an ω-compatible complex structure $J : TX \to TX$; that is, $J^2 = -1$ and $g(v, w) = \omega(v, Jw)$ is a Riemannian metric. Another way of saying this last thing is that $g(v, Jv) > 0$ for $v \neq 0$ and $\omega(Jv, Jw) = \omega(v, w)$. There is no obstruction to finding such almost complex structures. Assume now that g comes from J as above. The almost complex structure gives a decomposition $T^*X \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$. Here

$$T^{1,0}X = \{\alpha : TX \otimes \mathbb{C} \to \mathbb{C} \mid \alpha(Jv) = i\alpha(v)\}, T^{0,1}X = \{\alpha : TX \otimes \mathbb{C} \to \mathbb{C} \mid \alpha(Jv) = -i\alpha(v)\}.$$

We have also $\Lambda^2 T^*X \otimes \mathbb{C} = T^{2,0}X \oplus T^{1,1}X \oplus T^{0,2}$, where $T^{2,0}$ are $(1,0)$ forms wedge $(1,0)$ forms and so on. We have the Nijenhuis tensor $N : T^{1,0} \to T^{0,2}$ given by $N(\alpha) = (d\alpha)^{0,2}$.

Exercise 97. This is a tensor.

Fact 98. *J is integrable (i.e. (X, J) is a complex manifold) if and only if $N = 0$.*

It is obvious that if (X, J) is complex, then $N = 0$; the other way is the Newlander–Niremberg theorem.

Recall from last time that ω is self-dual with respect to the metric, and $|\omega| = \sqrt{2}$. At any point, we can find local coordinates such that $\omega = dx_0dx_1 + dx_2dx_3$, $J(\frac{\partial}{\partial x_0}) = \frac{\partial}{\partial x_1}$, $J(\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial x_3}$, and $g_{ij} = \delta_{ij}$. We also saw that $\mathrm{cl}(\omega) = S_+ \to S_+$ has eigenvalues $\pm 2i$, and we let E be the $-2i$ eigenspace. The map $\mathfrak{s} \mapsto E$ gives a map from the set of spin-c structures $\mathrm{Spin}^c(X)$ on X to the set of line bundles on X, which equals $H^2(X;\mathbb{Z})$. This map is actually an equivariant (with respect to the action of $H^2(X;\mathbb{Z})$ – remember that $\mathrm{Spin}^c(X)$ is affine over $H^2(X;\mathbb{Z})$) bijection, and with this identification, we can view the Seiberg–Witten invariant as a map $\mathrm{SW}(X, \cdot) = H^2(X;\mathbb{Z}) \to \mathbb{Z}$. Generally, this map depends on the homology orientation on X; in the symplectic case there is a canonical homology orientation.

Theorem 99 (Taubes)**.** *We have the following*

- *$SW(X, 0) = 1$.*
- *$SW(X, K) = \pm 1$, where $K = T^{2,0}X$.*
- *If $SW(x, e) \neq 0$, then $0 \leq [\omega] \cdot e \leq [\omega] \cdot K$.*
- *If $SW(x, e) = 0$, then $e = 0$ or $e = K$.*

This is just the starting point of the more general "Seiberg–Witten = Gromov" theorem, which says that $\mathrm{SW}(X, e)$ equals a certain count of J-holomorphic curves representing the class $PD(e) \in H_2(X)$. If we are willing to believe this, the above theorem is obvious.

Proof of theorem. We follow `http://math.berkeley.edu/~hutching/pub/tn.pdf`. The idea is the following: Instead of the small perturbation μ ensuring the transversality as before, we take instead μ to be a large multiple of ω. This will deform the Seiberg–Witten equations a lot, and ultimately the zero set of the E component of ψ will turn into a J-holomorphic curve. We will not try to understand this part, but we will see the calculations leading to the theorem.

Lemma 100. *Set $K = T^{2,0}$, $K^{-1} = T^{0,2}$. There are isomorphims $S_+ = E \oplus K^{-1}E = (T^{0,0} \otimes E) \oplus (T^{0,2} \otimes E)$, $S_- = T^{0,1} \otimes E$ such that if $v \in T^*X \otimes \mathbb{C}$ and $\alpha \in T^{0,*} \otimes E$, then*

$$cl(v) \cdot \alpha = \sqrt{2}(v^{0,1} \wedge \alpha - 1(\overline{v^{1,0}})\alpha).$$

Proof. As before, E is the $-2i$ eigenspace of $\mathrm{cl}(\omega)$. We have an isometry $\frac{1}{2}\mathrm{cl} : K^{-1} \otimes E \to$ ($+2i$ eigenspace of $\mathrm{cl}(\omega)$). This can be seen by just checking it, as we have

$$\mathrm{cl}(\frac{\partial}{\partial x_0}) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \mathrm{cl}(\frac{\partial}{\partial x_i}) = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}.$$

$K^{-1} = T^{0,2}$ is spanned over $\mathbb{C}$ by $(dx_0 - idx_1) \wedge (dx_2 - idx_3)$. We have

$$\begin{aligned}\mathrm{cl}((dx_0 - idx_1) \wedge (dx_2 - idx_3)) &= \mathrm{cl}(dx_0 - idx_1)\mathrm{cl}(dx_2 - idx_3) \\ &= \begin{pmatrix} 0 & -I - i\sigma_1 \\ I - i\sigma_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_2 - i\sigma_3 \\ \sigma_2 - i\sigma_3 & 0 \end{pmatrix}.\end{aligned}$$

On S_+ this is

$$(-I - i\sigma_1)(\sigma_2 - i\sigma_3) = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix}.$$

This gives us the desired isomorphism, and putting in the $\frac{1}{2}$, we actually that $\frac{1}{2}\mathrm{cl}$ is an isometry.

We also have an isometry $\frac{1}{\sqrt{2}}\mathrm{cl} : T^{0,1} \otimes E \to S_-$, which can also just be checked. With these identification, we simply check that Clifford multiplication is given as claimed.

□

Note that in general, any complex manifold in any dimension has a spin structure, and the above lemma basically defines it.

We have $S_+ = E \oplus K^{-1}E$, and the conjugate spin-c structure gives $\overline{S_+} = E^{-1} \oplus KE^{-1} = KE^{-1} \oplus K^{-1}(KE^{-1})$; that is, conjugation of spin-c structures sends E to KE^{-1}, or in other words, in sends $e \mapsto c_1(K) - e$. This explains how the second part of the theorem follows from the first part by charge conjugation.

Let us now write the Dirac equation in a nice way.

Lemma 101. *There is a unique connection (sometimes called the Taubes connection) A_0 on K^{-1} such that if $E = \underline{\mathbb{C}}$, where $\underline{\mathbb{C}}$ denotes the trivial line bundle, and if $u_0 = 1 \in \Gamma(E)$, then $\nabla_{A_0} u_0 \in \Gamma(T^*X \otimes K^{-1})$ and $D_{A_0} u_0 = 0$.*

Proof. Let A be any connection on $K^{-1} = \det(\mathfrak{s})$. Then

$$\nabla_A u_0 = (a, b) \in \Gamma(T^*X \otimes (\underline{\mathbb{C}} \oplus K^{-1})).$$

Because the spin connection is compatible with the Hermitian metric on the spin bundle, we have $a \in i\Omega^1$. This follows from

$$0 = d\langle u_0, u_0 \rangle = \langle \nabla_A u_0, u_0 \rangle + \langle u_0, \nabla_A u_0 \rangle = a + \bar{a}.$$

Take $A_0 = A - 2a$. Now $\nabla_{A_0} u_0 \in \Gamma(T^*X \otimes K^{-1})$, since subtracting $2a$ has the effect of subtracting a in the expression for ∇_A.

So, $\nabla_{A_0} u_0 = (0, b) \in \Gamma(T^*X \otimes K^{-1})$. We now have the most confusing identity ever,

$$D_{A_0}(\mathrm{cl}(\omega) u_0) = \mathrm{cl}((d + d^*)\omega) u_0 + \mathrm{cl}(\omega \cdot \nabla_{A_0} u_0),$$

for any differential form ω, any spinor u_0, and any connection A_0. This is like the Leibniz rule for the Dirac operator; the $d + d^*$ is the Dirac operator for forms, which explains the first term. In the second term, the $\cdot$ denotes Clifford multiplication by ω, and we then sort of do Clifford multiplication on anything left. The equality follows from the fact that $\nabla_{A_0}(\alpha \cdot \psi) = \nabla \alpha \cdot \psi + \alpha \nabla_{A_0} \psi$, which is left as an exercise.

The identity tells us that in our case,

$$-2iD_{A_0} u_0 = \mathrm{cl}(2i\nabla_{A_0} u_0) = 2iD_{A_0} u_0,$$

since $d\omega = 0$, and by self-duality, $d^*\omega = 0$. In other words, $D_{A_0} u_0 = 0$.

Note that we also have the uniqueness claimed in the theorem. □

In the general case, $\det(\mathfrak{s}) = K^{-1} \otimes E^2$. Write the connection A on $\det(\mathfrak{s})$ as $A = A_0 + 2a$, where a is a connection on E.

Lemma 102. $D_A\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \sqrt{2}(\overline{\partial}_a\alpha + \overline{\partial}_a^*\beta)$. *Here, remember that* $\nabla_a = \Gamma(E) \to \Gamma(T^*X \otimes E)$, *and* $\overline{\partial}_a$ *is the* $T^{0,1} \otimes E = S_-$ *component of* ∇_a.

So now we understand the Dirac equation, and we move on to the curvature equation

$$F_A^+ = \sigma(\psi, \psi) + i\mu.$$

Write $\psi = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \Gamma(E \oplus K^{-1}E)$. We claim that the curvature equation comes out to be the following:

$$F_A^+ = i(|\alpha|^2 - |\beta|^2)\omega + 2(\overline{\alpha}\beta - \alpha\overline{\beta}) + i\mu.$$

Here $\overline{\alpha}\beta$ is a section of K^{-1} and $\alpha\overline{\beta}$ a section of K. This makes sense since $\Lambda_+^2 \otimes \mathbb{C} = \mathbb{C}\omega \oplus K \oplus K^{-1}$.

Taubes' perturbation is the following: Put $\mu = -r\omega - iF_{A_0}$, where $r \gg 0$. Observe that if $E = \underline{\mathbb{C}}$, there is now a "trivial solution" where $A = A_0$, so $a = 0$, with $\alpha = \sqrt{r}u_0$, $\beta = 0$. The curvature equation then says

$$F_{A_0}^+ = F_A^+ = ir\omega + i\mu = ir\omega - ir\omega + F_{A_0}^+,$$

which is true. It turns out that this trivial solution corresponds to the count of J-holomorphic curves, and a calculation shows that the trivial solution is unique up to gauge equivalence for $r \gg 0$. We continue the proof next time.

18th lecture, October 28 2010

So, we're trying to prove the following.

Theorem 103. *Let* (X^4, ω) *be a symplectic manifold with* $b_2^+ > 1$ *so that* $Spin^c(X) \cong H^2(X;\mathbb{Z})$.

- *If* $SW(X,e) \neq 0$, *then* $e \cdot [\omega] \geq 0$
- $SW(X, 0) = 1$.
- *If* $SW(X,e) \neq 0$ *and* $e \cdot [\omega] = 0$ *then* $e = 0$.

Ultimately, $\mathrm{SW}(X, E)$ counts certain holomorphic curves dual to E.

The first part of the proof is to rewrite the Seiberg–Witten equations in terms of symplectic geometry. The second part is to perturb using ω. The third and final part is some clever estimates.

Choose ω-compatible J and identify $S_+ = \Lambda^{0,\mathrm{even}} \otimes E$, $S_- = \Lambda^{0,1} \otimes E$. With these identifications $\mathrm{cl}(v) \cdot \alpha = \sqrt{2}(v^{0,1} \wedge \alpha - 1(\overline{v^{1,0}})\alpha)$ for $v \in T^*X \otimes \mathbb{C}$, $\alpha \in S$.

There is a canonical connection A_0 on K^{-1} such that if $E = \underline{\mathbb{C}}$ then $D_{A_0}(u_0) = 0$, where here $u_0 \in \Gamma(\underline{\mathbb{C}})$ is constantly equal to one.

In the general case, we can write the connection A on $\det(\mathfrak{s}) = E^2 \otimes K^{-1}$ as $A_0 + 2a$ where a is a connection on E.

Lemma 104. $D_{A_0+2a}(\alpha, \beta) = \sqrt{2}(\overline{\partial_a}\alpha + \overline{\partial_a}^*\beta)$, *for* $\alpha \in \Gamma(E)$, $\beta \in \Gamma(K^{-1}(E))$.

Proof. It is enough to prove this for $E = \underline{\mathbb{C}}$, and $a = 0$, and the general case follows by tensoring. In this case

$$\begin{aligned}
D_{A_0}(\alpha + \beta) &= D_{A_0}((\alpha + \beta/2) \cdot u_0) \\
&= [(d + d^*)(\alpha + \beta/2)] \cdot u_0 \\
&= d\alpha \cdot u_0 + \frac{d\beta + d^*\beta}{2} \cdot u_0 \\
&= \sqrt{2}\overline{\partial}\alpha + d^*\beta \cdot u_0 \\
&= \sqrt{2}\overline{\partial} + \sqrt{2}\overline{dd}^*\beta,
\end{aligned}$$

where the $\frac{1}{2}$ in the first equality comes from our consideration of $\frac{1}{2}\mathrm{cl}$ from last time, and the second equality follows from our calculation of the Dirac operator from last time. In the fourth one, we use that $d^*\beta = *d*\beta = *d\beta$, which holds since β is self-dual. $\square$

We turn now to (Taubes' version of) the curvature equation

$$\frac{1}{4}(F_A^+ - i\mu) = \sigma(\psi, \psi).$$

(Note the difference from last time.) We will compute $\sigma(\psi,\psi)$, which we claim to be $\frac{i}{4}(|\alpha|^2 - |\beta|^2)\omega + \frac{1}{2}(\overline{\alpha}\beta - \alpha\overline{\beta})$.

Recall the local model

$$\mathrm{cl}(e_0) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \mathrm{cl}(e_i) = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}.$$

Clifford multiplication defines a map $\mathrm{cl} : i\Lambda^2_+ \to \mathfrak{su}(S_+)$, and then $\sigma(\psi,\psi) = \mathrm{cl}^{-1}((\psi\psi^*)_0)$. We calculate

$$\begin{aligned}
\mathrm{cl}(e_0e_1 + e_2e_3) &= \begin{pmatrix} -2\sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \\
\mathrm{cl}(e_0e_2 - e_1e_3) &= \begin{pmatrix} -2\sigma_2 & 0 \\ 0 & 0 \end{pmatrix} \\
\mathrm{cl}(e_0e_3 + e_1e_2) &= \begin{pmatrix} -2\sigma_3 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

In this local model, with $\psi = \begin{pmatrix} a \\ b \end{pmatrix}$, we have

$$\begin{aligned}
(\psi\psi^*)_0 &= \begin{pmatrix} \frac{1}{2}(|a|^2 - |b|^2) & a\overline{b} \\ \overline{a}b & \frac{1}{2}(|b|^2 - |a|^2) \end{pmatrix} \\
&= \frac{-i}{2}(|a|^2 - |b|^2)\sigma_1 + \frac{1}{2}(\overline{a}b - a\overline{b})\sigma_2 - \frac{i}{2}(\overline{a}b + a\overline{b})\sigma_3.
\end{aligned}$$

We find that

$$\begin{aligned}
\sigma(\psi,\psi) =& \frac{i}{4}(|a|^2 - |b|^2)(dx_0dx_1 + dx_2dx_3) - \frac{1}{4}(\overline{a}b - a\overline{b})(dx_0dx_2 - dx_1dx_3) \\
&+ \frac{i}{4}(\overline{a}b + a\overline{b})(dx_0dx_3 + dx_1dx_2).
\end{aligned}$$

Write $\alpha = a$ and, by a calculation from last time,

$$\begin{aligned}
\beta &= \frac{-b}{2}(dx_0 - idx_1)(dx_2 - idx_3) \\
&= \frac{-b}{2}(dx_0dx_2 - dx_1dx_3) + \frac{ib}{2}(dx_0dx_3 + dx_1dx_2).
\end{aligned}$$

Since $\omega = dx_0dx_1 + dx_2dx_3$, we get

$$\sigma(\psi,\psi) = \frac{i}{4}(|\alpha|^2 - |\beta|^2)\omega + \frac{1}{2}(\overline{\alpha}\beta - \alpha\overline{\beta}).$$

The curvature equation becomes

$$F_{A_0}^+ 2F_A^+ = i(|\alpha|^2 - |\beta|^2)\omega + 2(\overline{\alpha}\beta - \alpha\overline{\beta}) + i\mu.$$

This completes the first part of the proof; rewriting the Seiberg–Witten equations: The second one becomes $\overline{\partial}_a\alpha = -\overline{\partial}_a^*\beta$ by the lemma.

Recall Taubes' perturbation: We put $\mu = -r\omega + iF^+_{A_0}$. Using this the curvature equation now becomes

$$F^+_a = -\frac{i}{2}(r - |\alpha|^2 + |\beta|^2)\omega + (\overline{\alpha}\beta - \alpha\overline{\beta}).$$

Rescale α and β by $\sqrt{r}$, and obtain

$$F^+_a = -\frac{ir}{2}(1 - |\alpha|^2 + |\beta|^2)\omega + r(\overline{\alpha}\beta - \alpha\overline{\beta}).$$

Now to the third part of the proof. Step 1: By the Dirac equation

$$\begin{aligned}\overline{\partial_a}\,\overline{\partial_a}^*\beta &= -\overline{\partial_a}\,\overline{\partial_a}\alpha \\ &= -F_a^{0,2}\alpha + N(\partial_a\alpha) \\ &= -r|\alpha|^2\beta + N(\partial_a\alpha).\end{aligned}$$

Here, we have used that

$$(F_a\alpha)^{0,2} = (\nabla_a^2\alpha)^{0,2} = (\nabla_a(\partial_a\alpha + \overline{\partial_a}\alpha))^{0,2} = N(\partial_a\alpha) + \overline{\partial_a}\,\overline{\partial_a}\alpha,$$

where $N : \Lambda^{1,0} \to \Lambda^{0,2}$ is $N(\eta) = (d\eta)^{0,2}$.

Now, we can integrate over the manifold and obtain

$$\int |\overline{\partial_a}^*\beta|^2 = \int (\langle N(\partial_a\alpha), \beta\rangle - r|\alpha|^2|\beta|^2)$$

The left hand side is $\int |\overline{\partial_a}\alpha|^2$. Since N is a tensor, $|N(\partial_a\alpha)| \leq c|\nabla_a\alpha|$. Thus

$$\langle N(\partial_a\alpha), \beta\rangle \leq \frac{c}{r}|\nabla_a\alpha|^2 + \frac{r}{2}|\beta|^2.$$

We obtain

$$2\int |\overline{\partial_a}\alpha|^2 \leq \int \frac{2c}{r}|\nabla_a\alpha|^2 + r|\beta|^2(1 - |\alpha|^2).$$

Step 2: By using a Weitzenböck formula for $\overline{\partial_a}$, we get

$$\begin{aligned}2\overline{\partial_a}^*\overline{\partial_a}\alpha &= \nabla_a^*\nabla_a\alpha - i\langle F_a, \omega\rangle\alpha \\ &= \nabla_a^*\nabla_a\alpha - r(1 - |\alpha|^2 + |\beta|^2)\alpha.\end{aligned}$$

We see that

$$2\int |\overline{\partial_a}\alpha|^2 = \int (|\nabla_a\alpha|^2 - r(1 - |\alpha|^2 + |\beta|^2)|\alpha|^2).$$

Step 3: Recall the following from Chern–Weil theory, that $[F_a] = -2\pi ie$, where $e = c_1(E)$. This tells us that

$$\begin{aligned}2\pi[\omega]\cdot e &= \int iF_a \wedge \omega = \int i\langle F_a, \omega\rangle \\ &= r(1 - |\alpha|^2 + |\beta|^2)\end{aligned}$$

Using step 2, we have

$$\int |\nabla_a\alpha|^2 = 2\pi[\omega]\cdot e + \int (2|\overline{\partial_a}|^2 + r(1 - |\alpha|^2 + |\beta|^2)(|\alpha|^2 - 1)),$$

and using step 1,

$$\int (1 - \frac{2c}{r})|\nabla_a \alpha|^2 + r(1 - |\alpha|^2)^2 \leq 2\pi[\omega] \cdot e.$$

Take $r > 2c$ such that everything is positive. It follows that if (a, α, β) is a solution, then the $[\omega] \cdot e \geq 0$. If $[\omega] \cdot e = 0$, then $\nabla_a \alpha = 0$, and $|\alpha| = 1$. The last equality implies that $E = \underline{\mathbb{C}}$, and we can apply a gauge transformation such that $a = 0$ and $\alpha = 1$. From the curvature equation, $\beta = 0$. Note that $E = \underline{\mathbb{C}}$, $\alpha = 1$, $\beta = 0$, $a = 0$ actually is a solution. By the equality above, this is the only solution. We have shown that if $[\omega] \cdot e = 0$, then $e = 0$, and the solution is unique up to gauge equivalence. This almost implies that $\mathrm{SW}(X, 0) = \pm 1$; we have to check transversality if $r \gg 0$: That the deformation complex at this point has $H^2 = 0$ (which is left as a (good) exercise). $\square$

19th lecture, November 2 2010

6.3 The adjunction inequality

Theorem 105 (The adjunction inequality). *Let X be a closed oriented manifold with $b_2^+(X) \geq 2$, let Σ be a (connected) oriented smoothly embedded surface of genus $g > 0$. Suppose $\Sigma \cdot \Sigma \geq 0$ and $SW(X, \mathfrak{s}) \neq 0$. Then $2g - 2 \geq \Sigma \cdot \Sigma + |c_1(\mathfrak{s}) \cdot \Sigma|$.*

Suppose X above is symplectic. Then by Taubes' theorem from last time, $\mathrm{SW}(X, K) = \pm 1$. Therefore

$$2g - 2 \geq \Sigma \cdot \Sigma + |c_1(K) \cdot \Sigma|.$$

Note that if Σ is J-holomorphic, then $TX|_\Sigma = T\Sigma \oplus N_\Sigma$ as complex vector bundles. Taking c_1 on both sides, we get $-c_1(K) \cdot \Sigma = 2 - 2g + \Sigma \cdot \Sigma$.

Corollary 106 (The Thom conjecture). *If X is symplectic with $b_2^+ \geq 2$ and Σ is J-holomorphic with $\Sigma \cdot \Sigma \geq 0$. Then Σ is genus minimizing in its homology class.*

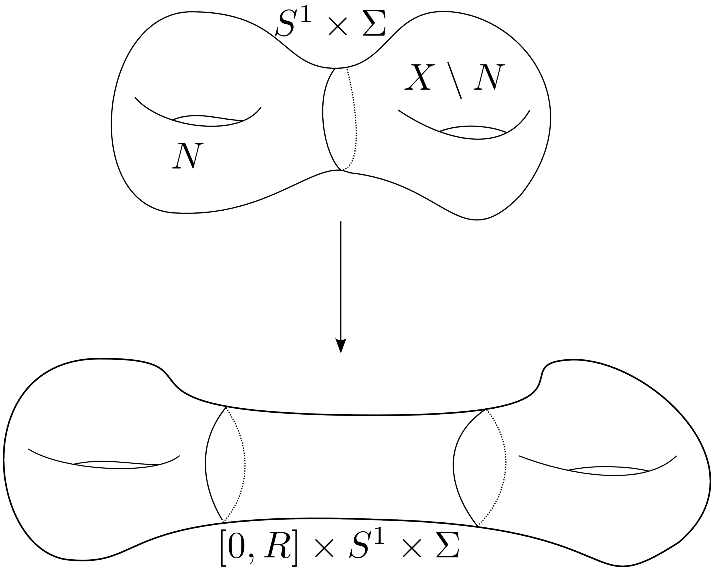

Figure 15: Stretching the neck of a surface.

Proof of theorem. The basic strategy to proof the adjunction inequality is the following:

1) Blow up to reduce to the case where $\Sigma \cdot \Sigma = 0$.

2) Write $X = N \cup_{S^1\times\Sigma} (X \setminus N)$ for N a tubular neighborhood of Σ. Stretch the neck as in Fig. 15, where the long piece has the product metric, to find an $\mathbb{R}$-invariant solution to the Seiberg–Witten equations on $\mathbb{R} \times S^1 \times \Sigma$.

3) Use Seiberg–Witten theory on $S^1 \times \Sigma$ to get the inequality.

We concentrate on step 2) for now.

Proposition 107. *Let X be a closed oriented smooth 4-manifold with $b_2^+(X) \geq 2$. Suppose $X = X_1 \cup_Y X_2$ for some hypersurface Y. Suppose $SW(X, \mathfrak{s}) \neq 0$. Then there exists an $\mathbb{R}$-invariant solution to the Seiberg–Witten equations on $\mathbb{R} \times Y$ for $\mathfrak{s}|_Y$.*

Proof. We need to talk about the Seiberg–Witten equations on $\mathbb{R} \times Y$. Denote the $\mathbb{R}$-coordinate by t. We have a spin-c structure $S_{\mathbb{R}\times Y} = S_+ \oplus S_- \to \mathbb{R} \times Y$ with Clifford multiplication $\mathrm{cl}(\partial_t) : S_+ \xrightarrow{\cong} S_-$, and we have a spin-c structure $(S_+|_{\{t\}\times Y}, \mathrm{cl}(\partial_t)^{-1}, \mathrm{cl}(\cdot))$ on Y; regard S_+ on $\mathbb{R} \times Y$ as pulled back from a bundle S on Y. If (A, ψ) is a solution to the Seiberg–Witten equations on $\mathbb{R} \times Y$, we can apply a gauge transformation to put A into *temporal gauge*, i.e.

$$\nabla_{A, \frac{\partial}{\partial t}} = \frac{\partial}{\partial t}.$$

That such a gauge transformation exists and is unique up to $\mathrm{Maps}(Y, S^1)$ is left as an exercise.

Assume that (A, ψ) is in temporal gauge. Write the Seiberg–Witten equations on $\mathbb{R} \times Y$ in terms of the data on Y. Write $(A, \psi) = \{(A(t), \psi(t))\}_{t\in\mathbb{R}}$ with $\psi(t) \in \Gamma(S)$, $A(t) \in \mathrm{Conn}(\det S)$. The Dirac operator behaves as

$$\begin{aligned} D_A\psi = \mathrm{cl}(\nabla_A\psi) &= \mathrm{cl}(dt \otimes \frac{\partial\psi}{\partial t} + \nabla_{A(t)}\psi(t)) \\ &= \frac{\partial\psi}{\partial t} + D_{A(t)}\psi(t). \end{aligned}$$

We claim that (A, ψ) satisfies the Seiberg–Witten equations on $\mathbb{R} \times Y$ if and only if

$$\begin{aligned} \frac{\partial\psi}{\partial t} &= -D_{A(t)}\psi(t) \\ \frac{\partial A}{\partial t} &= - *_3 F_{A(t)} + \tau(\psi(t)), \end{aligned}$$

where $\tau : S \to T^*Y$ is a quadratic map defined as $\tau(\psi) = \langle \mathrm{cl}(\cdot)\psi, \psi\rangle$. We've seen where the first equation comes from and now look at the curvature equation. We have

$$\begin{aligned} F_A &= F_{A(t)} + dt \wedge \frac{\partial A}{\partial t}, \\ *_4F_A &= dt \wedge *_3F_{A(t)} + *\frac{\partial A}{\partial t}. \end{aligned}$$

This implies that

$$F_A^+ = \frac{1}{2}(F_A + *_4F_A) = \frac{1}{2}dt \wedge \left(\frac{\partial A}{\partial t} + *_3F_{A(t)}\right) + \frac{1}{2}\left(*_3\frac{\partial A}{\partial t} + F_{A(t)}\right).$$

Clifford multiplication looks like

$$\begin{aligned} \mathrm{cl}(\partial_t) &= \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \mathrm{cl}(\partial_1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ \mathrm{cl}(\partial_2) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{cl}(\partial_3) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ \mathrm{cl}(\partial_2\partial_3) &= \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \end{aligned}$$

where we have local coordinates x_1, x_2, x_3 on Y. So $\mathrm{cl}(*_3\alpha) = -\mathrm{cl}(\alpha)$, if α is a 1-form or 2-form on Y. We have

$$\begin{aligned}\mathrm{cl}_4(F_A^+) &= \frac{1}{2}\left(-\mathrm{cl}_3(\frac{\partial A}{\partial t}) - \mathrm{cl}_3(*_3F_{A(t)}) - \mathrm{cl}_3(\frac{\partial A}{\partial t}) - \mathrm{cl}_3(*_3F_{A(t)})\right)\\ &= -\mathrm{cl}_3\left(\frac{\partial A}{\partial t} + *3F_{A(t)}\right).\end{aligned}$$

Recall that $\frac{1}{2}\mathrm{cl}_4(F_A^+) = (\psi\psi^*)_0$. That is,

$$-\frac{1}{2}\mathrm{cl}_3(\frac{\partial A}{\partial t} + *_3F_A(t)) = (\psi\psi^*)_0.$$

We need that $(\psi\psi^*)_0 = -\frac{1}{2}\mathrm{cl}_3(\tau(\psi))$. Note that if $\psi = \begin{pmatrix} a \\ b\end{pmatrix}$, we get

$$\begin{aligned}\tau(\psi) &= \langle \mathrm{cl}(\cdot)\psi, \psi\rangle\\ &= i(|a|^2 - |b|^2)dx_1 + (a\bar{b} - \bar{a}b)dx_2 + i(a\bar{b} + \bar{a}b)dx_3.\end{aligned}$$

Then the last 3D equation follows, since

$$\begin{aligned}\mathrm{cl}_3(\tau(\psi)) &= \begin{pmatrix} -(|a|^2 - |b|^2) & * \\ * & -(|b|^2 - |a|^2)\end{pmatrix}\\ &= -2(\psi\psi^*)_0.\end{aligned}$$

We claim now that the solutions to the Seiberg–Witten equations on $\mathbb{R}\times Y$ are gradient flowlines of a functional $F : \mathrm{Conn}(\det S_Y) \times \Gamma(S_Y) \to \mathbb{R}$, where S_Y is the bundle over Y (denoted S before). I.e. the path $t \mapsto (A(t), \psi(t))$ is a gradient flowline of F. Pick a reference connection A_0 on $\det S$. Define

$$F(A, \psi) = -\frac{1}{8}\int_Y (A - A_0) \wedge (F_A + F_{A_0}) + \frac{1}{2}\int_Y \langle D_A\psi, \psi\rangle d\mathrm{vol}.$$

Here, the first term is sometimes called the Chern–Simons term. We will calculate the differential of F. Let $\dot{A} \in i\Omega^1(Y)$, $\dot{\psi} \in \Gamma(S)$. We get that

$$\begin{aligned}dF_{(A,\psi)}(\dot{A}, \dot{\psi}) =& -\frac{1}{8}\int_Y \dot{A} \wedge (F_A + F_{A_0}) - \frac{1}{8}\int_Y (A - A_0) \wedge d\dot{A}\\ &+ \frac{1}{4}\int \langle \mathrm{cl}(\dot{A})\psi, \psi\rangle d\mathrm{vol} + \frac{1}{2}\int_Y (\langle D_A\dot{\psi}, \psi\rangle + \langle D_A\psi, \dot{\psi}\rangle)d\mathrm{vol}\\ =& -\frac{1}{4}\int \dot{A} \wedge (F_A - *\tau(\psi)) + \int_Y \mathrm{Re}\langle D_A\psi, \dot{\psi}\rangle d\mathrm{vol}\\ =& -\frac{1}{4}\int -\langle \dot{A}, *F_A - \tau(\psi)\rangle d\mathrm{vol} + \int_Y \mathrm{Re}\langle D_A\psi, \dot{\psi}\rangle d\mathrm{vol}\\ =& -\frac{1}{4}\langle \dot{A}, *F_A - \tau(\psi)\rangle_{L^2} + \langle D_A\psi, \dot{\psi}\rangle_{L^2}\end{aligned}$$

Up to some rescaling, these are the gradient flow equations of F. We continue from here next time.

20th lecture, November 4 2010

For a manifold Y^3 with spin-c structure $\mathfrak{s}$, we defined $F : \mathrm{Conn}(\det(\mathfrak{s})) \times \Gamma(S) \to \mathbb{R}$ by

$$F(A, \psi) = -\frac{1}{8}\int_Y (A - A_0) \wedge (F_A + F_{A_0}) + \frac{1}{2}\int_Y \langle D_A\psi, \psi\rangle.$$

As always, integration of a function means with respect to the volume form. The gradient flow lines of F are

$$\frac{\partial \psi(t)}{\partial t} = -D_{A(t)}\psi(t)$$
$$\frac{\partial A(t)}{\partial t} = - *_3 F_{A(t)} + \tau(\psi(t)).$$

Here, A_0 is a reference connection on $\det(\mathfrak{s})$, $\tau(\psi) = \langle \mathrm{cl}(\dot{)}\psi, \psi\rangle$and we consider a one-parameter family $\mathbb{R} \to \mathrm{Conn}(\det(\mathfrak{s})) \times \Gamma(S)$, $t \mapsto (A(t), \psi(t))$. Last time, we had a factor of 4. In Kronheimer–Mrowka they use another L^2 inner product, which doesn't really matter. The important thing is that the above are the Seiberg–Witten equations on $\mathbb{R} \times Y$ when $\nabla_A = \frac{\partial}{\partial t} + \nabla_{A(t)}$ is in temporal gauge. The critical points of F are pairs (A, ψ) satisfying $D_A\psi = 0$ and $*F_A = \tau(\psi)$. These we will call the 3-dimensional Seiberg–Witten equations. We were trying to prove the following:

Proposition 108. *If $b_2^+(X) \geq 2$, and $X = X_1 \cup_Y X_2$ satisfies $SW(X, \mathfrak{s}) \neq 0$, then there exists a $\mathbb{R}$-invariant solution to the Seiberg–Witten equations on $(\mathbb{R} \times Y, \mathfrak{s}|_Y)$.*

The strategy of the proof is to use neck stretching as in Fig. 15 – this corresponds to a changing of a metric in the neighborhood of Y. To prove the proposition we have to talk about *energy.* Let X^4 be a compact oriented Riemannian manifold with boundary Y^3 oriented with the boundary orientation, and let S be a spin-c structure on Y. Let $(A, \psi) \in \mathrm{Conn}(\det(\mathfrak{s})) \times \Gamma(\S_+)$. The idea now is to write down a variational form of the Seiberg–Witten equations; we want solutions of the equations to minimize some sort of energy. We define the *analytical energy*

$$\mathcal{E}^{an}(A, \psi) = \frac{1}{4}\int_X |F_A|^2 + \int_X |\nabla_A\psi|^2 + \frac{1}{4}\int_X (|\psi|^2 + (\frac{s}{2}))^2 - \int_X \frac{s^2}{16},$$

where s is scalar curvature. We define the *topological energy*

$$\mathcal{E}^{top}(A, \psi) = \frac{1}{4}\int_X F_A \wedge F_A - \int_Y \langle \psi, D_B\psi\rangle + \int_Y \frac{H}{2}|\psi|^2,$$

where D_B is the 3-dimensional Dirac operator $B = A|_Y$, and H is the mean curvature of Y in X (which will usually be 0 in our calculations).

Lemma 109. *$\mathcal{E}^{an}(A, \psi) \geq \mathcal{E}^{top}(A, \psi)$ with equality if and only if (A, ψ) satisfiees the Seiberg–Witten equations.*

Proof. We start out with the BLW formula

$$D_A^2\psi = \nabla_A^*\nabla_A\psi + \frac{1}{2}\mathrm{cl}(F_A^+)\psi + \frac{s}{4}\psi.$$

This implies that

$$\int_X \langle \psi, D_A^2\psi\rangle = \int \langle \psi, \nabla_A^*\nabla_A\psi\rangle + \frac{1}{2}\int_Y \langle \psi, \mathrm{cl}(F_A^+)\psi\rangle + \frac{1}{4}\int_X s|\psi|^2.$$

We would like to move the D_A and ∇_A to the other sides of the inner products, but here we have to take care of the boundary terms. We have that

$$\int_X \langle \psi, \nabla_A^*\nabla_A\psi\rangle = \int_X |\nabla_A\psi|^2 - \int_Y \langle \psi, \nabla_{A,\vec{n}}\psi\rangle,$$

where $\nabla_{A,\vec{n}}$ denotes the covariant derivative in the outwards pointing normal direction – the equality is left as an exercise. On the other hand

$$\int_X \langle \psi, D_A^2\psi\rangle = \int_X |D_A\psi|^2 - \int_Y \langle \psi, \mathrm{cl}(\vec{n})^{-1}D_A\psi\rangle.$$

This is an exercise as well. In the local model,

$$\mathrm{cl}_4(\vec{n}) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \mathrm{cl}_4(v) = \begin{pmatrix} 0 & \mathrm{cl}_3(v) \\ \mathrm{cl}_3(v) & 0 \end{pmatrix},$$

for $v \in TY$. We have

$$D_A(\psi|_Y) = \mathrm{cl}(\vec{n})^{-1}(D_A\psi)|_Y - (\nabla_{A,\vec{n}}\psi)|_Y + (H/2)\psi|_Y.$$

Putting these four formulas together, we get

$$\begin{aligned}\int_X |D_A\psi|^2 = \int_X |\nabla_A\psi|^2 + \frac{1}{2}\int_X \langle\psi, \mathrm{cl}(F_A^+)\psi\rangle \\ + \frac{1}{4}\int_X s|\psi|^2 + \int_Y \langle\psi, D_B\psi\rangle - \int_Y (H/2)|\psi|^2.\end{aligned}$$

The left hand side is 0 if and only if the first of the Seiberg–Witten equations is satisfied.

$$\int_X |\frac{1}{2}\mathrm{cl}(F_A^+) - (\psi\psi^*)_0|^2 = \frac{1}{2}\int_X |F_A^+|^2 + \frac{1}{4}\int_X |\psi|^2 - \frac{1}{2}\int_X \langle\psi, \mathrm{cl}(F_A^+)\psi\rangle.$$

Here, the left hand side is greater than or equal to 0 with equality if and only if the second Seibert–Witten equation is satisfied. We can write this as

$$\int_X |\frac{1}{2}\mathrm{cl}(F_A^+) - (\psi\psi^*)_0|^2 = \frac{1}{4}\int_X |F_A|^2 - \frac{1}{4}\int_X F_A \wedge F_A + \frac{1}{4}\int_X |\psi|^2 - \frac{1}{2}\int_X \langle\psi, \mathrm{cl}(F_A^+)\psi\rangle.$$

This follows from the equalities $F_A = F_A^+ + F_A^-$ and $F_A \wedge F_A = F_A^+ \wedge F_A^+ + F_A^- \wedge F_A^-$, which implies that $|F_A|^2 = |F_A^+|^2 + |F_A^-|^2$. If α is an imaginary 2-form, then $\int|\alpha|^2 = -\int \alpha \wedge *\alpha$. This implies that $\int F_A \wedge F_A - \int |F_A|^2 = 2\int F_A^+ \wedge F_A^+ = -2\int |F_A^+|^2$, which implies the above. Adding the previous equations, we obtain

$$\int_X (|D_A\psi|^2 + |\frac{1}{2}\mathrm{cl}(F_A^+) - (\psi\psi^*)_0|^2) = \mathcal{E}^{an}(A,\psi) - \mathcal{E}^{top}(A,\psi).$$

□

Lemma 110. *Suppose (A,ψ) is a solution on a cylinder $[t_1,t_2]\times Y$ (with the product metric) in temporal gauge, then $\mathcal{E}^{top}(A,\psi) = 2(F(A(t_1),\psi(t_1)) - F(A(t_2),\psi(t_2)))$. That is, the topological energy measures the change of the functional.*

Proof. Use the upward normal to orient both of the boundary components of the cylinder. Note that $H = 0$ in this case. We have

$$\mathcal{E}^{top}(A,\psi) = \frac{1}{4}\int_{[t_1,t_2]\times Y} F_A \wedge F_A + \int_Y |\psi(t_1), D_{A(t_1)}\psi(t_1)\rangle - \int_Y \langle\psi(t_2), D_{A(t_2)}\psi(t_2)\rangle.$$

Use the same reference connection A_0 on the top and the bottom of the cylinder. We need that

$$\begin{aligned}-\frac{1}{4}\int_Y (A(t_1) - A_0) \wedge (F_{A(t_1)} + F_{A_0}) + \frac{1}{4}\int_Y (A(t_2) - A_0) \wedge (F_{A(t_2)} + F_{A_0}) \\ = \frac{1}{4}\int_{[t_1,t_2]\times Y} F_A \wedge F_A.\end{aligned}$$

This follows from Stokes' theorem since

$$d((A - A_0) \wedge (F_A + F_{A_0})) = (F_A - F_{A_0}) \wedge (F_A + F_{A_0}) = F_A \wedge F_A$$

□

We can now finish the proof of the proposition. Let (A_R, ψ_R) be a solution on the neck-stretched manifold. We claim that there exists an R-independent upper bound on $F(A_R(0), \psi_R(0)) - F(A_R(R), \psi_R(R))$, where here we consider (A_R, ψ_R) as functions of the cylinder coordinate. This expression equals

$$\begin{aligned}
\mathcal{E}^{top}(A_R, \psi_R)|_{[0,R]\times Y} &= \mathcal{E}^{top}(A_R, \psi_R) - \mathcal{E}^{top}(A_R, \psi_R)|_{X_1} - \mathcal{E}^{top}(A_R, \psi_R)|_{X_2}. \\
&= -\pi^2 c_1(\mathfrak{s})^2 - \mathcal{E}^{an}(A_R, \psi_R)|_{X_1} - \mathcal{E}^{an}(A_R, \psi_R)|_{X_2} \\
&\leq -\pi^2 c_1(\mathfrak{s})^2 + \int_{X_1} \frac{s^2}{16} + \int_{X_2} \frac{s^2}{16},
\end{aligned}$$

which is independent of R, proving the claim. For all n there exists R and a subinterval of the form $[0,1] \times Y$ of $[0,R] \times Y$ such that the energi of this subinterval is less than $1/n$. By a general compactness result, there exists a solution on $[0,1] \times Y$ with energy equal to 0, which implies that it is $\mathbb{R}$-invariant. □

Next time, we will see how this proposition proves the adjunction inequality. □

21st lecture, 16th November 2010

Last time we considered neck-stretching; cutting a 4-manifold X with $b_2^+(X) \geq 2$ into three pieces X_1, X_2, and $[0,R] \times Y$ as in Fig. 15 and considering $R \to \infty$.

Proposition 111. *If $SW(X, \mathfrak{s}) \neq 0$ then there exists an $\mathbb{R}$-invariant solution to the Seiberg–Witten equations on $\mathbb{R} \times Y$.*

We recall the proof. The Seiberg–Witten solutions are characterized by $\mathcal{E}^{top}(A, \psi) = \mathcal{E}^{an}(A, \psi)$, where

$$\begin{aligned}
\mathcal{E}^{top}(A, \psi) &= \frac{1}{4}\int_X F_A \wedge F_A - \int_Y \langle \psi, D_b \psi \rangle + \int_Y (H/2)|\psi|^2, \\
\mathcal{E}^{an}(A, \psi) &= \frac{1}{4}\int_X |F_A|^2 - \int_X |\nabla_A \psi|^2 + \frac{1}{4}\int (|\psi|^2 + (s/2))^2 - \int_X s^2/16.
\end{aligned}$$

Here X is any oriented 4-manifold with oriented boundary Y. On $[t_1, t_2] \times Y$, we have

$$\mathcal{E}^{top}(A, \psi) = 2(F(t_1) - F(t_2)),$$

if (A, ψ) is in tempoeral gauge; $\nabla_{A, \frac{\partial}{\partial t}} = \frac{\partial}{\partial t}$. Here $F : \mathrm{Conn}(\det(S_Y)) \times \Gamma(S_Y) \to \mathbb{R}$ is given by

$$F(A, \psi) = -\frac{1}{8}\int_Y (A - A_0) \wedge (F_A + F_{A_0}) + \frac{1}{2}\int_Y \langle D_A \psi, \psi \rangle.$$

Claim 112. *On the neck-streched manifold, there is an R-independent upper bound on $F(0) - F(R)$.*

Proof. For any Seiberg–Witten solution, we have

$$\begin{aligned}
2(F(0) - F(R)) &= \mathcal{E}^{top}(A, \psi)|_{[0,R]\times Y} \\
&= \mathcal{E}^{top}(A, \psi)_X - \mathcal{E}^{top}(A, \psi)_{X_1} - \mathcal{E}^{top}(A, \psi)_{X_2} \\
&= -2\pi i c_1(\mathfrak{s})^2 - \mathcal{E}^{an}(A, \psi)_{X_1} - \mathcal{E}^{an}(A, \psi)_{X_2},
\end{aligned}$$

which is less than or equal to some constant. □

We have the following general compactness theorem, which we won't prove.

Theorem 113 ([KM]). *If X is compact with boundary with spin-c structure $\mathfrak{s}$, then any sequence in $\mathcal{M}(X, \mathfrak{s})$ with $\mathcal{E} \leq c$ has a convergent subsequence (in C^∞) – here $\mathcal{E}$ is either the topological or analytical energy (as they are equal).*

Take $R \to \infty$. There then exists a sequence (A_n, ψ_n) of solutions on $[0,1] \times Y$ with energy $\mathcal{E}(A_n, \psi_n) < 1/n$. By the compactness theorem, we can pass to a subsequence converging to (A, ψ) on $[0,1] \times Y$ with energy $\mathcal{E}(A, \psi) = 0$, and $\mathcal{E}(A, \psi) = F(0) - F(1)$, so this is an $\mathbb{R}$-invariant solution on $[0,1] \times Y$. This ended the proof of the proposition.

We will use the proposition to prove the following.

Theorem 114 (The adjunction inequality). *If $b_2^+(X) \geq 2$, $SW(X, \mathfrak{s}) \neq 0$, and $\Sigma \hookrightarrow X$ is a closed oriented connected surface with genus $g(\Sigma) > 0$, and $\Sigma \cdot \Sigma \geq 0$, then*

$$2g - 2 \geq \Sigma \cdot \Sigma + |c_1(\mathfrak{s}) \cdot \Sigma|.$$

Proof. We have a blow-up formula, allowing us to reduce to the case where $\Sigma \cdot \Sigma = 0$. Using that $\Sigma \cdot \Sigma = 0$, we can consider a tubular neighborhood of the surface, $N(\Sigma) \cong \Sigma \times D^2$, and write $X = N(\Sigma) \cup_{\Sigma \times S^1} X \setminus N(\Sigma)$. By the proposition, there exists an $\mathbb{R}$-invariant solution to the Seiberg–Witten equations on $\mathbb{R} \times S^1 \times \Sigma$, i.e. to the 3-dimensional Seiberg–Witten equations on $S^1 \times \Sigma$: $D_A \psi = 0$ and $*F_A = \tau(\psi, \psi)$.

Recall the BLW formula

$$D_A^2 = \nabla_A^* \nabla_A \psi - \frac{1}{2} \mathrm{cl}_3(*F_A)\psi + \frac{s}{4}\psi.$$

The same argument as before that $|\psi|^2 \leq \max_Y(-\frac{s}{2}, 0)$. We take the product metric on $S^1 \times \Sigma$, where Σ has unit area and constant scalar curvature (equal to twice the Gauss curvature), such that $-\frac{s}{2} = 2\pi(2g-2)$ (which is possible as $g(\Sigma) > 0$. We have the upper bound

$$|\psi|^2 \leq 2\pi(2g-2),$$

so $|F_A| \leq 2\pi(2g-2)$. Now

$$\begin{aligned} [F_A] &= -2\pi i c_1(\mathfrak{s}|_{S^1 \times \Sigma}) \\ &= -2\pi i c_1(\mathfrak{s}) \cdot \Sigma, \\ |c_1(\mathfrak{s}) \cdot \Sigma| &= |-\frac{1}{2\pi i} \int_\Sigma F_A| \leq \frac{1}{2\pi} \sup(|F|) \operatorname{area}(\Sigma) \\ &\leq 2g - 2. \end{aligned}$$

□

7 3-dimensional Seiberg–Witten theory

The simple cartoon picture of 3-dimensional Seiberg–Witten theory is the following: We would like to make Seiberg–Witten theory into a topological quantum field theory. For X^4 closed, we associate to $(X^4, \mathfrak{s})$ the integer $\mathrm{SW}(X, \mathfrak{s})$. For Y^3 closed, we would like to associate to $(Y^3, \mathfrak{s})$ a vector space, denoted by $HM_*(Y, \mathfrak{s})$. This will be the Morse homology (or the Floer homology) of the functional F (there are several different versions of this, as we will see later). Given a compact 4-manifold X with boundary Y, we would like to associate an element $Z(X, \mathfrak{s}) \in HM_*(Y, \mathfrak{s})$. The way to do this is to consider instead $\overline{X}$ given by attaching a cylinder $[0, \infty) \times Y$ to X as in Fig. 16. Then $Z(X, \mathfrak{s})$ counts finite energy solution (in the 0-dimensional moduli spaces) on $\overline{X}$; any finite energy solution on $\overline{X}$ gives a solution to the Seiberg–Witten equations on Y, and so we get an element of the chain complex for HM_*. We look at 1-dimensional moduli spaces on $\overline{X}$ – the boundary of this space is $\partial Z(X, \mathfrak{s}) = 0$, so $Z(X, \mathfrak{s})$ is actually a cycle, so $Z(X, \mathfrak{s}) \in HM_*(Y, \mathfrak{s})$. Similar arguments show that $Z(X, \mathfrak{s})$ is independent of the choices made. Switching the orientation on Y, we get $HM_*(-Y, \mathfrak{s}) = HM^*(Y, \mathfrak{s})$. If X has boundary components $-Y_0$ and Y_1, then $Z(X, \mathfrak{s}) \in \operatorname{Hom}(HM_*(Y_0, \mathfrak{s}), HM_*(Y_1, \mathfrak{s}))$. Considering $X = X_1 \cup_{Y_1} X_2$, where X_1 has boundary components Y_0, Y_1 and X_2 has boundary components Y_1, Y_2 (Fig. 17), then we have a gluing

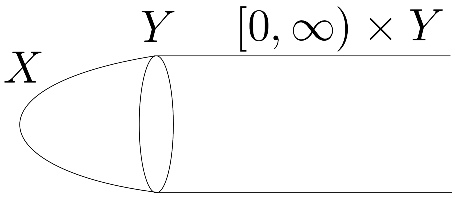

Figure 16: The manifold $\overline{X}$.

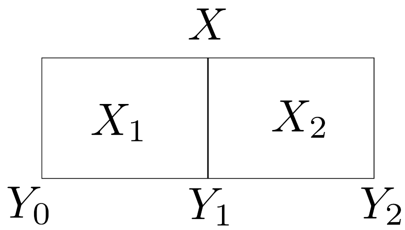

Figure 17: The gluing of two 4-manifolds.

$$Z(X,\mathfrak{s}) = Z(X_2,\mathfrak{s}) \circ Z(X_1,\mathfrak{s}) : HM_*(Y_0,\mathfrak{s}) \to HM_*(Y_2,\mathfrak{s}).$$

This is proved by using neck-stretching, inserting a large enough cylinder $[0,R]\times Y_1$ between X_1 and X_2.

The above is only a cartoon picture and it does not work as simple in real life:

1) The functional F is not gauge invariant, making it hard to define HM_* in terms of F – this turns out to not be so bad, and we will consider this problem next time.

2) The grading on HM_* is ambiguous – this isn't too bad either.

3) The moduli space of gradient flow lines are not cut out transversally for generic pertubations on Y – there are hundreds of pages in [KM] to deal with this. They introduce a more general way to perturb the equations; we will not consider this problem.

4) The last and most serious problem involves the reducibles. Different ways of dealing with the irreducibles leads to different versions of HM. We will also see how these work out next time.

22nd lecture, November 18th 2010

Let Y be a closed connected 3-manifold, $\mathfrak{s}\in \mathrm{Spin}^c(Y)$ a spin-c structure, $F:\mathrm{Conn}(\det(\mathfrak{s}))\times\Gamma(S)\to\mathbb{R}$ our functional from before,

$$-\frac{1}{8}\int_Y (A-A_0)\wedge(F_A+F_{A_0}) + \frac{1}{2}\int_Y \langle D_A\psi,\psi\rangle.$$

We have seen that Seiberg–Witten solutions on $\mathbb{R}\times Y$ in temporal gauge correspond to gradient flowlines of F. We want to define some kind of "Morse homology" of F. The idea is to define a chian complex with generators the critical points of F, $\mathrm{Crit}(F)$, which correspond to solutions to the 3d Seiberg Witten equations on Y. The differential should count flow lines between critical points of "index difference one". As we saw last time, there are several issues:

1) Failure of gauge invariance of F: Remember that a map $g:Y\to S^1$ acts as $g\cdot(A,\psi) = (A-2g^{-1}dg, g\psi)$. Then

$$F(g\cdot(A,\psi)) - F(A,\psi) = -\frac{1}{8}\int_Y -2g^{-1}dg\wedge(F_A+F_{A_0}) + \frac{1}{2}\int_Y\langle D_{A-2g^{-1}dg}g\psi, g\psi\rangle - \frac{1}{2}\int_Y\langle D_A\psi,\psi\rangle.$$

Here, $D_{A-2g^{-1}dg}g\psi = gD_A\psi$, so the last two integrals cancel. Recall that we have an identification $[Y, S^1] = H^1(Y;\mathbb{Z})$ mapping $g \mapsto [\frac{1}{2\pi i}g^{-1}dg]$. This means that $[g^{-1}dg] = 2\pi i\phi(g)$, and $[F_A] = [F_{A_0}] = -2\pi i c_1(\mathfrak{s})$. Putting all of this together,

$$F(g\cdot(A,\psi)) - F(A,\psi) = 2\pi^2\phi(g)\cdot c_1(\mathfrak{s}).$$

The gauge group has different connected components, $\pi_0\mathcal{G} = H^1(Y;\mathbb{Z})$, and F is invariant under the identitiy component, $\mathcal{G}_0$, of $\mathcal{G}$. It is tempting to simply take the quotient with the identity component. We get a covering space

$$\begin{tikzcd} H^1(Y;\mathbb{Z}) \arrow[r] & \mathcal{C}/\mathcal{G}_0 \arrow[r, "F"] \arrow[d] & \mathbb{R} \\ & \mathcal{C}/\mathcal{G} & \end{tikzcd}$$

and dF is a well-defined closed 1-form on $\mathcal{C}/\mathcal{G}$.

More generally, one could consider the Morse theory of closed 1-forms in finite dimension: Let X be a closed smooth manifold with a metric g and let α be a closed 1-form on X. On the universal covering $\tilde{X}$, there exists f as below

$$\begin{tikzcd} H_1(X) \arrow[r] & \tilde{X} \arrow[r, "f"] \arrow[d] & \mathbb{R} \\ & X & \end{tikzcd}$$

with $df = \pi^*\alpha$. We can define Morse homology of $f : \tilde{X} \to \mathbb{R}$ as a module over the "Nonkov ring". We don't have to do this in the Seiberg–Witten case though, and we don't go into details.

2) The index of a critical point is not (obviously) defined: The Hessian of a critical point has infinitely many negative eigenvalues and infinitely many positive eigenvalues. However, a flow line γ from p to q has a well-defined index. In the finite-dimensional model, we have a map $\gamma : \mathbb{R} \to X$ satisfying $\frac{d\gamma}{dt} + \nabla f(\gamma(t)) = 0$, $\lim_{t\to-\infty}\gamma(t) = p$, $\lim_{t\to\infty}\gamma(t) = q$. We want to understand the dimension of the moduli space $\mathcal{M}(p,q)$. The tangent space $T\mathcal{M}(p,q)$ is the kernel of the operator $D : \Gamma(\gamma^*TX) \to \Gamma(\gamma^*TX)$, where we impose decay conditions on the sections at $\pm\infty$ – e.g. one could use $L^2_1(\mathbb{R}, \gamma^*TX)$, so one could consider $D : L^2_1(\mathbb{R}, \gamma^*TX) \to L^2(\mathbb{R}, \gamma^*TX)$. Then $D : L^2_1(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n)$ has the form $D = \frac{\partial}{\partial t} + A(t)$, where $A(t)$ is an $n \times n$ matrix. It turns out that D is Fredholm, and that the Morse–Smale transversality condition is equivalent to D being surjective. Then $\dim\mathcal{M}(p,q) = \operatorname{ind}(D)$.

Fact 115. *Assume that* $\lim_{t\to\pm\infty} A(t) \in M_n$ *exists and is symmetric and invertible. Then* D *is Fredholm and* ind(D) *is equal to the spectral flow of the family* $\{A_t\}$*, i.e. the number of eigenvalues that cross from negative to positive as* t *goes from* $-\infty$ *to* $+\infty$*. This is nothing but the difference between the number of positive eigenvalues of* $\lim_{t\to\infty} A(t)$ *and positive eigenvalues of* $\lim_{t\to-\infty} A(t)$*. (Exercise: Check this for* $n = 1$*.)*

In our Morse theory situation, the limits will just be the Hessians, $\lim_{s\to-\infty} A(t) = H(f,p)$ and $\lim_{s\to\infty} A(t) = H(f,q)$, so the spectral flow is $(n - \operatorname{ind}(q)) - (n - \operatorname{ind}(p)) = \operatorname{ind}(p) - \operatorname{ind}(q)$. The above fact generalizes to the infinite-dimensional case, and allows us to come up with a definition of the index. Now consider the functional F, and suppose $\gamma = \{(A_t, \psi_t)\}_{t\in\mathbb{R}}$ is a downward flow line of F from (A_-, ψ_-) to (A_+, ψ_+). The linearization of the 4d Seiberg–Witten equations at γ defines an operator of the form $D = \frac{\partial}{\partial t} + A(t)$ for some operators $A(t)$. If transversality holds, i.e. this operator D is surjective, then $\dim(\mathcal{M})$ near γ is nothing but $\operatorname{ind}(D)$, which is the spectral flow of $\{A_t\}$. The upshot is that $\operatorname{ind}(\gamma)$ is a well-defined integer.

We should be aware of the following: The index difference $\operatorname{ind}(A_-, \psi_-) - \operatorname{ind}(A_+, \psi_+)$ is not well-defined. It depends on the homotopy class of γ: Two different flow lines might have different indices; e.g. a loop γ might have non-trivial spectral flow – this never happens in the finite-dimensional case. If γ and γ' are two flow lines from (A_-, ψ_-) to (A_+, ψ_+), then $\gamma \cdot \overline{\gamma'} \in \pi_1(\mathcal{C}/\mathcal{G})$, which maps to $H^1(Y;\mathbb{Z})$, under the lift of the covering of $\mathcal{C}/\mathcal{G}$ described above.

Claim 116. *We have*

$$\mathit{ind}(\gamma) - \mathit{ind}(\gamma') = \pm c_1(\mathfrak{s}) \cdot \Phi(\gamma\overline{\gamma'}),$$

where Φ is the map $\pi_1(\mathcal{C}/\mathcal{G}) \to H^1(Y;\mathbb{Z})$ is induced by the covering as described above.

Sketch of proof. The index is "additive under gluing"; if X is obtained from X_1 and X_2 as in Fig. 17, then the index of X is the sum of of the indices for X_1 and X_2. We want to use this fact for two copies of $I \times Y$ glued to form a $S^1 \times Y$. Now, $\mathrm{ind}(\gamma) - \mathrm{ind}(\gamma')$ equals the dimension of the Seiberg–Witten moduli space on $S^1 \times Y$ for the spin-c structure obtained from gluing via the temporal gauges for γ and γ'; in other words, we glue via gauge transformation in the class $\Phi(\gamma\overline{\gamma'})$. Then the dimension is

$$\frac{c_1(\mathfrak{s}_{S^1\times Y})^2 - 2\chi(S^1 \times Y) - 3\sigma(S^1 \times Y)}{4} = \frac{c_1(\mathfrak{s}_{S^1\times Y})}{4}.$$

To calculate $c_1(\mathfrak{s}_{S^1\times Y})$, we take a generic section and see where it vanishes. Let $\pi : S^1 \times Y \to Y$ be the projection. Then

$$\begin{aligned} c_1(\mathfrak{s}_{S^1\times Y}) &= \pi^* c_1(\mathfrak{s}_Y) \pm 2[S^1] \smile \pi^*\Phi(\gamma\overline{\gamma'}), \\ c_1(\mathfrak{s}_{S^1\times Y})^2 &= \pm 4 c_1(\mathfrak{s}_Y) \cdot \Phi(\gamma\overline{\gamma'}). \end{aligned}$$

□

To sum up, we have

$$\begin{aligned} F(g \cdot (A,\psi)) - F(A,\psi) &= 2\pi^2 \phi(g) \cdot c_1(\mathfrak{s}) \\ \mathrm{ind}(g \cdot (A,\psi)) - \mathrm{ind}(A,\psi) &= \pm\phi(g) \cdot c_1(\mathfrak{s}). \end{aligned}$$

So, our two ambiguities actually cancel out: Let p and q be critical points. Then $\langle \partial p, q\rangle$ counts the index 1 flow lines from p to q – these all have the same energy, which is enough to ensure compactness.

23rd lecture, November 23rd 2010

7.1 Seiberg–Witten Floer homology

Let Y be a closed oriented connected 3-manifold, let $\mathfrak{s}$ be a spin-c structure on Y, and consider the function $F : \mathrm{Conn}(\det \mathfrak{s}) \times \Gamma(S) \to \mathbb{R}$ given by

$$F(A,\psi) = -\frac{1}{8}\int_Y (A - A_0) \wedge (F_A + F_{A_0}) + \frac{1}{2}\int_Y \langle D_A\psi, \psi\rangle.$$

We want to define the Seiberg–Witten–Floer homology to be the Morse homology of this functional. So recall that the critical points of F are solutions to $D_A\psi = 0$, $*F_A = \tau(\psi)$, and gradient flow lines are solutions to the 4-dimensional Seiberg–Witten equations on $\mathbb{R} \times Y$. Suppose that $c_1(\mathfrak{s}) \in H^2(Y;\mathbb{Z})$ is not torsion, so that there are no reducibles (recall that reducible solutions are those with $\psi \equiv 0$, $F_A \equiv 0$). In this case define Seiberg–Witten Floer homology $HM(Y, \mathfrak{s})$ as follows. (First off, make perturbations to make everything is transverse – this is a long story we won't go into.) Define a chain complex $CM(Y, \mathfrak{s})$ to be the free $\mathbb{Z}$-module generated by solutions to the 3-dimensional Seiberg–Witten equations modulo gauge equivalence. If α_-, α_+ are two generators, then the differential coefficient $\langle\partial\alpha_-, \alpha_+\rangle \in \mathbb{Z}$ is a (signed, which we won't discuss) count of gradient flowlines $\{(A_t, \psi_t) \mid t \in \mathbb{R}\}$ (modulo 3-dimensional gauge equivalence) with $[\lim_{t\to\pm\infty}(A_t, \psi_t)] = \alpha_\pm$ and index equal to 1 (and recall that the index is given by spectral flow). For given α_- and α_+ all the index 1 flow lines have the same energy, and so we can use the compactness theoerem implying that the coefficient $\langle\partial\alpha, \alpha_+\rangle$ is finite. A standard gluing argument implyes that $\partial^2 = 0$, invariant of (Y, S).

Thus $HM(Y, \mathfrak{s})$ has a relative $\mathbb{Z}/d$ grading where d is the divisibility of $c_1(\mathfrak{s})$ in $H^2(Y; \mathbb{Z})/\text{Torsion}$. Note that d is always even: TY is trivial, so there exists $\mathfrak{s}_0$ with $c_1(\mathfrak{s}_0) = 0$, and if $aH^2(Y, \mathbb{Z})$ then $c_1(\mathfrak{s}_0 + a) = 2a$, so $c_1(\mathfrak{s})$ is always divisible by 2.

So what if $c_1(\mathfrak{s})$ is torsion, so that there exist reducibles? In this case $\mathcal{B} = \mathcal{C}/\mathcal{G}$ is not a manifold. Pick a point $y \in Y$ and define $\mathcal{G}_y = \{g \in \mathcal{G} \mid g(y) = 1\}$. In this case we have a fiber bundle

$$\begin{tikzcd} S^1 \arrow[r] & \hat{\mathcal{B}} = \mathcal{C}/\mathcal{G}_y \arrow[d] \\ & \mathcal{B} = \mathcal{C}/\mathcal{G} \end{tikzcd}$$

Here $\mathcal{B} = \hat{\mathcal{B}}/S^1$, and $\hat{\mathcal{B}}$ *is* a manifold, since $\mathcal{G}_y$ acts freely on $\mathcal{C}$: The gauge action is $g(A, \psi) = (A - 2g^{-1}dg, g\psi)$ which equals (A, ψ) if and only if $dg = 0$, and $g\psi = \psi$.

7.2 The finite dimensional model

Let X be a closed smooth manifold, and assume that S^1 acts on X (with no finite stabilizers). Let $f : X \to \mathbb{R}$ be an S^1-invariant function. We want to understand Morse homology of f. The approach of Kronheimer and Mrowka has two steps:

1) First, blow up X along the fixed point set. Here, if $Z \subseteq X$, the blow up is given as follows: Take the partially defined exponential map exp from $\{v \in N_Z \mid |v| < \epsilon\}$ to a neighborhood of Z. Here N_Z is the normal bundle to Z. Now, replace the neighborhood with $\{(v, s) \mid v \in N_Z, |v| = 1, s \in [0, \epsilon)\}$, which maps to the neighborhood of Z by $(v, s) \mapsto \exp(sv)$. In other words, if Z is a point, it gets replaced by a sphere contained in the boundary of the new manifold. Denote the blow up by $\mathrm{Bl}(X, Z)$. It has boundary $\partial\,\mathrm{Bl}(X, Z) = -\{v \in N_Z \mid |v| = 1\}$, and we get a map $\mathrm{Bl}(X, Z) \to X$ which is a diffeomorphism away form Z. Denote by $\mathcal{C}^\sigma(X, \mathfrak{s})$ the blow up $\{(A, \psi, s) \mid \|\psi\|_{L^2} = 1, s \geq 0\}$. This maps to $\mathcal{C}(X, \mathfrak{s})$ by $(A, \psi, s) \mapsto (A, s\psi)$.

2) They then do Morse homology for compact manifolds with boundary, which is our next topic.

7.2.1 Morse homology for manifolds with boundary

If X is a compact manifold with boundary $\partial X = Y_+ - Y_-$, and $f : X \to \mathbb{R}$ a Morse function, such that $-\nabla f$ points in along Y_+ and out along Y_-, and one usually defines $H_*^{\text{Morse}}(f) = H_*(X, Y_-)$. Unfortunately, we can't use this, as instead we want to consider gradient flow lines tangent to the boundary of X. In this situation there are 3 kinds of critical points.

1) There are critical points in $\mathrm{int}(X)$.

2) There are boundary stable critical points on the boundary of X. Boundary stable means that flow lines can come from $\mathrm{int}\, X$ but not the other way around. In this case $\mathrm{ind}_X = \mathrm{ind}_{\partial X}$.

3) There are boundary unstable critical points, for which flow lines can flow into the manifold. Here, $\mathrm{ind}_X = \mathrm{ind}_{\partial X} + 1$.

There are 8 operators counting flow lines (with signs): There are those counting flow lines in ∂X: $\overline{\partial}^s_s, \overline{\partial}^s_u, \overline{\partial}^u_s, \overline{\partial}^u_u$. For example, $\overline{\partial}^s_u$ counts flow lines from stable to unstable points within the boundary. Similarly, there are ∂^0_0 counting lines from the interior to the interior, ∂^0_s counting lines from the interior to a boundary stable critical point, ∂^u_0 counting lines from a boundary unstable point to the interior and ∂_s^u counts lines from boundary unstable points to boundary stable, through the interior of X.

We want to make a differential out of these operators. We have 8 identities from index 2 moduli spaces. Define

$$\overline{\partial} = \begin{pmatrix} \overline{\partial}_s^s & \overline{\partial}_s^u \\ \overline{\partial}_u^s & \overline{\partial}_u^u \end{pmatrix}.$$

Then $\overline{\partial}^2 = 0$. We also have

$$-\partial_0^0\partial_0^0 + \partial_0^u\overline{\partial}_u^s\partial_s^0 = 0,$$

as we can picture as in Fig. 18; note that the index difference between the points in the boundary is 1, but in X they have index difference 1. Similarly,

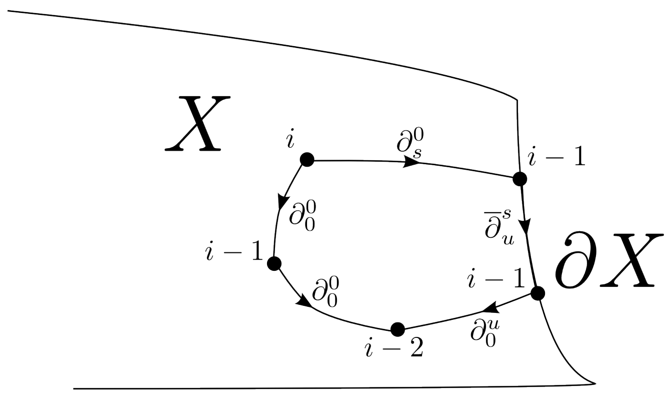

Figure 18: 5 critical points in a surface, illustrated with their indices, various paths between them, and the operators involved in the counts.

$$\begin{aligned} -\partial_s\partial^0 + \overline{\partial}_s^s\partial_s^0 + \partial_s^u\overline{\partial}_u^s\partial_s^0 &= 0, \\ -\partial_0^0\partial_0^u + \partial_0^u\overline{\partial}_u^u + \partial_0^u\overline{\partial}_u^s\partial_s^u &= 0, \\ -\overline{\partial}_s^u - \partial_s^0\partial_0^u - \overline{\partial}_s^s\partial_s^u 0\partial_s^u\overline{\partial}_u^u + \partial_s^u\overline{\partial}_u^s\partial_s^u &= 0. \end{aligned}$$

The next part of the approach of Kronheimer and Mrowka is to define three different chain complexes.

The first one is $\overline{C} = C^s \oplus C^u$ with differential $\overline{\partial}$ defined above.

The next one is $\check{C} = C^0 \oplus C^s$ with differential

$$\check{\partial} = \begin{pmatrix} \partial_0^0 & -\partial_0^u\overline{\partial}_u^s \\ \partial_s^0\overline{\partial}_s^s - \partial_s^u\overline{\partial}_u^s & \end{pmatrix}.$$

The last one is $\hat{C} = C^0 \oplus C^u$ with differential

$$\hat{\partial} = \begin{pmatrix} \partial_0^0 & \partial_0^u \\ -\overline{\partial}_u^s\partial_s^0 & -\overline{d}_u^u - \overline{\partial}_u^s\partial_s^u \end{pmatrix}.$$

Proposition 117. *These are chain complexes and have homologies*

$$\begin{aligned} \overline{H}_* &= H_*(\partial X), \\ \check{H}_* &= H_*(X), \\ \hat{H}_* &= H_*(X, \partial X). \end{aligned}$$

Furthermore, we can give a Morse-theoretic definition of a long exact sequence

$$\cdots \to H_*(X) \to H_*(X, \partial X) \to H_{*-1}(\partial X) \to \cdots$$

Parts of the proof. Let us first prove that $\check{\partial}^2 = 0$. We have

$$\check{\partial}^2 = \begin{pmatrix} \partial_0^0\partial_0^0 - \partial_0^u\overline{\partial}_u^s\partial_s^0 & -\partial_0^0\partial_0^u\overline{\partial}_u^s - \partial_0^u\overline{\partial}_u^s\overline{\partial}_s^s + \partial_0^u\overline{\partial}_u^s\partial_s^u\overline{\partial}_u^s \\ \cdots & \cdots \end{pmatrix}.$$

The first entry is 0 by the first identity above. Similarly, one could check the other entry by combining various identities and using that $\overline{\partial}^2 = 0$, and one can check the other chain complex.

To see the second part of the proposition, notice that the homologies in question only depend on X. Choose f such that all boundary critical points are boundary unstable. In this case, $\check{C} = C^0$, and $\check{H} = H_*(X)$ by using standard Morse homology with boundary. Similarly, one can treat $\hat{H}$. $\square$

In the Seiberg–Witten context,

$$\cdots \to H\check{M} * (Y, \mathfrak{s}) \to \overline{HM}_{*-1}(Y, \mathfrak{s}) \to H\hat{M}_{*-1}(Y, \mathfrak{s}) \to \cdots.$$

Here $\overline{HM}_{*-1}(Y, \mathfrak{s})$ is determined by the boundary and therefore by reducibles.

24th lecture, November 30 2010

Let Y^3 be a closed oriented 3-manifold with a spin-c structure $\mathfrak{s}$. We have a functional F : $\mathrm{Conn}(\det(\mathfrak{s})) \times \Gamma(S) \to \mathbb{R}$ given by

$$F(A, \psi) = -\frac{1}{8}\int_Y (A - A_0) \wedge (F_A + F_{A_0}) + \frac{1}{2}\int_Y \langle D_A\psi, \psi\rangle.$$

We want to define the Morse homology of this functional. Consider $\mathcal{B} = \mathcal{C}/\mathcal{G}$, where $\mathcal{G} = \mathrm{Maps}(Y, S^1)$ acts as $g \cdot A(\psi) = (A - 2g^{-1}dg, g\psi)$. We have

$$F(g \cdot (A, \psi)) - F(A, \psi) = 2\pi^2 c_1(\mathfrak{s}) \cdot [g] H^1(Y; \mathbb{Z}),$$

. so F is not quite gauge invariant, but we are able to define the index difference of a flow, since also

$$\mathrm{ind}(g \cdot (A, \psi)) - \mathrm{ind}(A, \psi) = \pm[g]c_1(\mathfrak{s}).$$

We will define the Morse homology of F on $\mathcal{B}$. After perturbation we have only finitely many critical points, and we have the necessary compactness. One problem is that $\mathcal{B}$ is not quite a manifold: $\mathcal{G}$ acts freely on $\mathcal{C}$ except on reducibles. Those instead have a S^1-stabilizer. The solution of Kronheimer–Mrowka is to first blow up along reducibles: Define the blow up

$$\mathcal{C}^\sigma = \{(A, \psi, s) \mid \|\psi\|_{L^2},\ s \geq 0\}$$

and a blow down map $\pi : \mathcal{C}^\sigma \to \mathcal{C}$ mapping $(A, \psi, s) \mapsto (A, s\psi)$ which is surjective and a diffeomorphism from the subset with $s \neq 0$ to irreducibles. Now $\mathcal{G}$ acts freely on the $\mathcal{C}^\sigma$ as

$$g(A, \psi, s) = (A - 2g^{-1}dg, g\psi, s).$$

Now $\mathcal{B}^\sigma = \mathcal{C}^\sigma/\mathcal{G}$ is a manifold with boundary. The gradient flow of F on $B \setminus \{reducibles\}$ extends to a (partially defined) vector field on $\mathcal{B}^\sigma$, tangent to the boundary.

We were considering the finite dimensional Morse theory for $f : X \to \mathbb{R}$, where X is compact with boundary, and ∇F is tangent to ∂X. In this case there are three kinds of critical points, interior (0), boundary stable (s) and boundary unstable (u). As last time, we have 8 operators counting flow lines (with signs) satisfying 8 identities coming from index 2 moduli spaces,

$$
\begin{aligned}
\overline{\partial}^2 = 0, \begin{pmatrix} \overline{\partial}^s_s & \overline{\partial}^u_s \\ \overline{\partial}^s_u & \overline{\partial}^u_u \end{pmatrix}, \\
-\partial^0_0 \partial^0_0 + \partial^u_0 \overline{\partial}^s_u \partial^0_s = 0 \\
-\partial^0_s \partial^0_0 - \overline{\partial}^s_s \overline{\partial}^0_s + \partial^u_s \overline{\partial}^s_u \partial^0_s = 0 \\
-\partial^0_0 \partial^u_0 0 \partial^u_0 \overline{\partial}^u_u + \partial^u_0 \overline{\partial}^s_u \partial^u_s = 0 \\
-\overline{\partial}^u_s - \partial^0_s \partial^u_0 - \overline{\partial}^s_s \partial^u_s + \partial^u_s \overline{\partial}^u_u + \partial^u_s \overline{\partial}^s_u \partial^u_s = 0.
\end{aligned}
$$

From this we can put together 3 chain complexes. One is $\overline{C} = C^s \oplus C^u$ with differential $\overline{\partial}$, and $H_*(\overline{C}, \overline{\partial}) = H_*(\partial X)$. Similarly, we have $\check{C} = C^0 \oplus C^s$ with differential

$$
\check{\partial} = \begin{pmatrix} \partial^0_0 & -\partial^u_0 \overline{\partial}^s_u \\ \partial^0_s & \overline{\partial}^s_s - \partial^u_s \overline{\partial}^s_u \end{pmatrix}.
$$

Proposition 118. *We have $H_*(\check{C}, \check{\partial}) = H_*(X)$.*

Idea of proof. First show that this is independent of f. Choose f such that all boundary critical points are boundary unstable. Then a standard theorem saying that we obtain the homology of X. □

Finally there's a complex $\hat{C} = C^0 \oplus C^u$ with differential

$$
\hat{\partial} = \begin{pmatrix} \partial^0_0 & \partial^u_0 \\ -\overline{\partial}^s_u \partial^0_s & -\overline{\partial}^u_u - \overline{\partial}^s_u \partial^u_s \end{pmatrix}.
$$

Fact 119. $H_*(\hat{C}, \hat{\partial}) = H_*(X, \partial X)$.

There is a long exact sequence

$$
\cdots \to \check{H}_* \xrightarrow{j_*} \hat{H}_* \xrightarrow{p_*} \overline{H_{*-1}} \xrightarrow{i_*} \check{H}_{*-1} \to \cdots.
$$

The chain map $j : \check{C}_* \to \hat{C}_*$, the anti-chain map $p : \hat{C}_* \to \overline{C}_{*-1}$, and the chain map $i : \overline{C}_* \to \check{C}_*$ are given by

$$
j = \begin{pmatrix} 1 & 0 \\ 0 & -\overline{\partial}^s_u \end{pmatrix}, p = \begin{pmatrix} \partial^0_s & \partial^u_s \\ 0 & 1 \end{pmatrix}, i = \begin{pmatrix} 0 & -\partial^u_0 \\ 1 & -\partial^u_s \end{pmatrix},
$$

satisfying $\hat{\partial} j = j \check{\partial}$, $\overline{\partial} p = -p \hat{\partial}$, and $\check{\partial} i = i \overline{\partial}$. The resulting long exact sequence agrees with the long exact sequence on homology,

$$
\cdots \to H_*(X) \to H_*(X, \partial X) \to H_{*-1}(\partial X) \to H_{*-1}(X) \to \cdots.
$$

7.3 Application to Seiberg–Witten theory

We now do an analogous construction for ∇f on $\mathcal{B}^\sigma$, and get three versions of Morse homology of F in a long exact sequence,

$$
\cdots \to H\check{M} * (Y, \mathfrak{s}) \to H\hat{M}_*(Y, \mathfrak{s}) \to \overline{HM}_{*-1}(Y, \mathfrak{s}) \to H\check{M}_{*-1}(Y, \mathfrak{s}) \to \cdots
$$

Remark 120. Notice first that $\overline{HM}_*(Y, \mathfrak{s})$ is determined by the reducibles. In particular, if $c_1(\mathfrak{s})$ is not torsion, then $\overline{HM}_*(Y, \mathfrak{s}) = 0$, and $H\check{M}_*(Y, \mathfrak{s}) = H\hat{M}_*(Y, \mathfrak{s})$ (and we denote the latter groups simply by $HM_*(Y, \mathfrak{s})$).

Theorem 121. *If $c_1(\mathfrak{s})$ is torsion, then $H\check{M}, H\hat{M}, \overline{HM}$ are all infinitely generated.*

The key ingredient of the proof of this theorem is the computation of $\overline{HM}$. We will return to this if time permits. Taubes uses this to prove the Weinstein conjecture.

Example 122. The simplest example is $Y = S^3$ with the unique spin-c structure $\mathfrak{s}$. There is a relative $\mathbb{Z}$ grading, and we can normalize to an absolute $\mathbb{Z}$ grading such that

$$\overline{HM}_* = \begin{cases} \mathbb{Z} & \text{if } * \text{ is even} \\ 0 & \text{if } * \text{ is odd} \end{cases}$$
$$\check{HM}_* = \begin{cases} \mathbb{Z} & \text{if } * = 0, 2, 4, \dots \\ 0 & \text{otherwise} \end{cases}$$
$$\hat{HM}_* = \begin{cases} \mathbb{Z} & \text{if } * = -1, -3, \dots \\ 0 & \text{otherwise} \end{cases}$$

In this example, we have a positive scalar curvature metric, so there are no irreducible solutions, and in fact there is a unique reducible solution up to gauge equivalence. In general, when $c_1(\mathfrak{s})$, the set of reducibles modulo gauge equivalence can be identified (non-canonically) with $H^1(Y; i\mathbb{R})/H^1(Y; i\mathbb{Z})$. The reason is the following: Reducibles correspond to flat connections on $\det(\mathfrak{s})$. Let A_0 be one of them (noting that one exists). Then $A_0 + a$ is another one if and only if $da = 0$, since $F_{A_0+a} = F_{A_0} + dA$. That means that the set of reducibles is an affine space over the set $E = \{a \in i\Omega^1(Y) \mid da = 0\}$ of closed imaginary 1-forms on Y. The set gauge transformations can be identified with $\{a \in i\Omega(Y) \mid da = 0, \frac{1}{2\pi}[a] \in H^1(Y; i\mathbb{Z})\}$. Even though in our case we have only one reducible solution, blowing up gives us infinitely many critical points of ∂B^σ. These can be ordered (see Fig. 19), and will give rise to the above homologies. To see this, we consider the finite dimensional model.

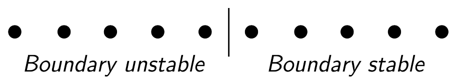

Figure 19: Ordering of critical points

Consider $f : \mathbb{C}^n \to \mathbb{R}$ defined by $f(z) = \frac{1}{2}\langle z, Lz\rangle$, where $L : \mathbb{C}^n \to \mathbb{C}^n$ is Hermitian. This is S^1-invariant with a unique critical point at 0. Assume L has distinct eigenvalues, all non-zero. We claim that if we blow up $\mathbb{C}^n$ at 0 and take the quotient with S^1 (to get $[0, \infty) \times \mathbb{C}P^{n-1}$), we get n critical points on the boundary: The critical points correspond to eigenvectors of L. Denote the eigenvalues of L by $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Boundary stable critical points correspond to those with $\lambda_i > 0$. The index of the critical point will be $2(i-1)$ if $\lambda_i > 0$, and $2(i-1)+1$ if $\lambda_i < 0$. (In the infinite dimensional setting, L has infinitely many positive and negative eigenvalues.) To prove the claim, consider $\Lambda : \mathbb{C}P^{n-1} \to \mathbb{R}$ defined by $\Lambda(z) = \frac{1}{2}\frac{\langle z, Lz\rangle}{\|z\|^2}$, $z \in \mathbb{C}^n \setminus \{0\}$. The blowup modulo S^1 is $\{(\phi, s) \mid \phi \in \mathbb{C}P^{n-1}, s \geq 0\}$. The downward gradient flow equations are $\dot{\phi} = -L\phi + \Lambda(\phi)\phi$, $\dot{s} = -\Lambda(\phi)s$ (where $L\phi$ means something particular). In $\mathbb{R}^2$, one can picture it as in Fig. 20.

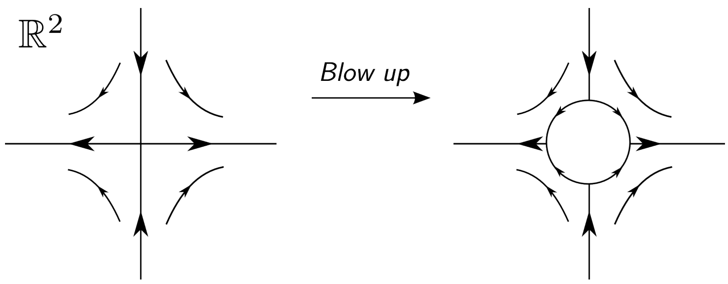

Figure 20: Blowing up at 0.

25th lecture, December 2 2010

7.4 The Weinstein conjecture

Today we give a very rough introduction to the Weinstein conjecture[3]. Let Y be a closed oriented 3-manifold.

Definition 123. A *contact form* on Y is a 1-form λ on Y such that $\lambda \wedge d\lambda > 0$ (everywhere). More generally, in $2n-1$ dimensions, a *contact form* is a 1-form λ such that $\lambda \wedge (d\lambda)^{n-1} > 0$.

Definition 124. An oriented hypersurface Y in a symplectic manifold (X, ω) is of *contact type* if there exists a contact form λ on Y with $d\lambda = \omega|_Y$.

Example 125. If $Y \subseteq \mathbb{R}^{2n}$ is the boundary of a star-shaped domain (meaning that Y is transverse to the gradient vector field), then Y is of contact type with contact form

$$\lambda = \frac{1}{2}\sum_{i=1}^{n}(x_i dy_i - y_i dx_i).$$

A contact form λ on Y^{2n-1} determines the *Reeb vector field* R characterized by $\lambda(R, \cdot) = 0$, normalized so that $\lambda(R) = 1$.

Note that if $Y \subseteq (X, \omega)$ is of contact type, and if Y is a regular level set of a Hamiltonian $H : X \to \mathbb{R}$, then on Y, the Hamiltonian vector field X_H is transverse to R.

Conjecture 126 (The Weinstein conjecture)**.** *If Y is a closed oriented odd-dimensional manifold with a contact form λ, then the associated Reeb vector field R has a closed orbit; that is, a trajectory of R starting and ending at the same point.*

This was first proved to be true on star-shaped domains in $\mathbb{R}^{2n}$. This motivated Weinstein to formulate the above conjecture in 1978. Today we will about the following theorem:

Theorem 127 (Taubes, 2006)**.** *The Weinstein conjecture is true in dimension 3.*

The conjecture is still open in higher dimensions, but partial results exist. Prior to Taubes' result, Hofer proved the following:

Theorem 128 (Hofer, 1993)**.** *It is true in dimension 3 if Kerλ is overtwisted (which we won't define), or if $\pi_2(Y) \neq 0$, or if $Y = S^3$. In these cases there exists a contractible Reeb orbit.*

Example 129. Consider the boundary ellipsoid $\partial E(A_1, a_2) \subseteq \mathbb{R}^4$, where $a_1, a_2 > 0$, and

$$\partial E(a_1, a_2) = \left\{(z_1, z_2) \in \mathbb{C}^2 \mid \frac{\pi|z_1|^2}{a_1} + \frac{\pi|z_2|^2}{a_2} = 1\right\}$$

Then the Reeb vector field is given by

$$R = 2\pi \sum_{i=1}^{2} a_i^{-1}\frac{\partial}{\partial\theta_i},$$

where

$$\frac{\partial}{\partial\theta_i} = x_i\frac{\partial}{\partial y_i} - y_i\frac{\partial}{\partial x_i}.$$

Assume a_i/a_2 is irrational. Then there are two embedded Reeb orbits given by $z_1 = 0$, $z_2 = 0$. We can also take teh quotient to get a contact form on the lens space $L(p, q)$ with two Reeb orbits. These are the only known examples with only finitely many Reeb orbits.

[3]A survey article by Hutchings is available online, [Hut]. On the MSRI website, from June 2008, there are videos of lectures on this topic.

Theorem 130. *If λ is non-degenerate (which basically means that the Reeb orbits are cut out transversally), and Y is not S^3 or a lens space, there exist more than 3 Reeb orbits.*

Theorem 131 (Colin–Honda). *There is a large family of examples in which there are infinitely many Reeb orbits.*

In the rest of this lecture, we consider the proof by Taubes. The motivation for it is the following: Recall that if (X^4, ω) is closed and symplectic, then $\mathrm{SW}(X)$ counts holomorphic curves in X. The first step in this is that if $S = E \oplus K^{-1}E$, $\psi = (\alpha, \beta)$, doing Taubes' perturbation and considering instead $r\omega$, then $\alpha^{-1}(0)$ converges to a holomorphic curve as $r \to \infty$.

If $\mathrm{SW}(X, \mathfrak{s}) \neq 0$, then there exists a holomorphic curve C with $[C] = PD(c_1(E))$. The 3-dimensional analogue is that if $HM(Y, \mathfrak{s}) \neq 0$ (for one of the versions of HM defined previously), then there exists a union of Reeb orbits representing Γ where $c_1(\mathfrak{s}) = c_1(\mathrm{Ker}\lambda) + 2PD(\Gamma)$. The idea is to do this 3-dimensional analogue and using the same perturbation that gave rise to holomorphic curves in 4 dimensions to get the Reeb orbits we are looking for.

Let $P(Y)$ be the set of oriented 2-plane fields (i.e. oriented rank 2 sub-vector bundles in the tangent bundle) on Y considered up to homotopy. The map $\pi : P(Y) \to \mathrm{Spin}^c(Y)$ given by $\xi \mapsto S = \underline{\mathbb{C}} \oplus \xi$, where Clifford multiplication is given as follows: If $y \in Y$, choose an oriented orthonormal basis $\{e_1, e_2, e_3\}$ for T_yY such that $\{e_2, e_3\}$ is an oriented basis for ξ_y. In the basis $\{1, e_2\}$ for S_y,

$$\mathrm{cl}(e_1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \mathrm{cl}(e_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathrm{cl}(e_3) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

This defines a spin-c structure.

Fact 132. *The map $\pi : P(Y) \to Spin^c(Y)$ is surjective. Given $\mathfrak{s} \in Spin^c(Y)$, $\pi^{-1}(\mathfrak{s})$ is an affine space over $\mathbb{Z}/d$, where d is the divisibility of $c_1(\mathfrak{s})$ in $H^2(Y;\mathbb{Z})/Torsion$. The various versions of $HM(Y, \mathfrak{s})$ have absolute gradings which associate to each generator an element of $\pi^{-1}(\mathfrak{s}) \subset P(Y)$.*

If λ is a contact form then $\xi = \mathrm{Ker}\lambda$ is an oriented 2-plane field, giving rise to a spin-c structure $\mathfrak{s}_\xi$. We get an identification $H_1(Y) \cong \mathrm{Spin}^c(Y)$ associating to Γ the spin-c structure $s_\xi + PD(\Gamma)$. The inverse of the above map is given as follows: Given a spin-c structure $\mathfrak{s}$, write $S = E \oplus K^{-1}E$, where E is the $+i$ eigenspace of $\mathrm{cl}(R)$, where $K^{-1} = \xi$. We then have $\Gamma = PD(c_1(E))$ and $\psi = (\alpha, \beta)$. We perturb the Seiberg–Witten equations using $r\lambda$ as $r \to \infty$, and the idea is that $\alpha^{-1}(0)$ should give us the orbits of the Weinstein conjecture.

Theorem 133 (Taubes). *Let (Y^3, λ) be a closed oriented contact manifold. Suppose $\widehat{HM}_*(Y, \mathfrak{s}) \neq 0$. If $E \cong \underline{\mathbb{C}}$ is the trivial bundle, we assume further that $\widehat{HM}_*(Y, \mathfrak{s}) \neq \mathbb{Z}$. Then there exists a non-empty finite set of Reeb orbits with total homology class $\Gamma = PD(c_1(E))$.*

Theorem 134 (Kronheimer–Mrowka). *If $c_1(\mathfrak{s})$, then $\widehat{HM}_*$, $\check{HM}_*$, and $\overline{HM}_*$ are all infinitely generated.*

By construction, $\widehat{HM}_*$ has grading bounded from above, and $\check{HM}_*$ has grading bounded from below. This follows from considerations like those from the last lecture. We have an exact sequence $\check{HM}_* \to \widehat{HM}_* \to \overline{HM}_* \to \cdots$. The hard part of the proof of the above theorem by Kronheimer and Mrowka then consists in showing that $\overline{HM}_*$ is infinitely generated with grading bounded in both directions. Note also that in this context, $c_1(\mathfrak{s}) = c_1(\xi) + 2c_1(E)$. We get the following corollary of the two theorems.

Corollary 135. *If $\Gamma \in H_1(Y)$, and if $c_1(\xi) + 2PD(\Gamma)$ is torsion, then there exists a non-empty finite set of Reeb orbits with total homology class Γ.*

Note that a class Γ satisfying the assumptions of the corollary always exists, because TY is trivial. This in particular implies the Weinstein conjecture in dimension 3.

The perturbed 3-dimensional Seiberg–Witten equations look as follows:

$$\begin{aligned} D_A\psi &= 0 \\ *F_A &= \rho(\psi) + i * \mu, \end{aligned}$$

where μ is an exact 2-form. Solutions are critical points of the perturbed functional

$$F(A,\psi) = F_{\text{usual}} + \frac{i}{4}\int_Y (A - A_0) \wedge \mu.$$

When μ is exact, F behaves under gauge transformation the same way F_{usual} did.

Taubes now chooses a metric with $|\lambda| = 1$, $d\lambda = 2 * \lambda$ (where in fact the factor of 2 is inconvenient) and defines.

$$\mu = -rd\lambda - iF_{A_0} + 2 * \overline{\omega},$$

where $[*\overline{\omega}] = \pi c_1(\xi)$. As in the 4-dimensional case, there exists a unique A_0 on ξ such that if $E = \underline{\mathbb{C}}$, then $D_{A_0}\psi_0 = 0$. A general connection on $\det(\mathfrak{s})$ is $A_0 + 2A$ for $A \in \mathrm{Conn}(E)$. With these conventions, the equations become

$$\begin{aligned} D_A\psi &= 0 \\ *F_A &= r(\rho(\psi) - i\lambda) + i\overline{\omega}, \end{aligned}$$

for $A \in \mathrm{Conn}(E)$. As $r \to \infty$, given a sequence of solutions $\psi_n = (\alpha_n, \beta_n)$, A_n, under favorable circumstances, $\alpha_n^{-1}(0)$ converges to a non-empty union of Reeb orbits. These favorable circumstances are the following:

1) There exists $\delta > 0$ with $\sup_Y(1 - |\psi_n|) > \delta$. This condition avoids the empty set.

2) There exists C with $i\int_Y \lambda \wedge F_{A_n} < C$, where this integral converges to the length of the union of Reeb orbits. This condition says that for $r \gg 0$ we have $\beta \sim 0$, $\alpha \sim 1$ except near $\alpha^{-1}(0)$, and F_A is concentrated near $\alpha^{-1}(0)$.

References

[Hut] Michael Hutchings. Taubes's proof of the Weinstein conjecture in dimension three. *Bulletin of the AMS*, 47:73–125, 2010.

[KM] Peter Kronheimer and Tomasz Mrowka. *Monopoles and three-manifolds*. New Mathematical Monographs, 2007.

[LM] H. Blaine Lawson and Marie-Louise Michelsohn. *Spin geometry*. Princeton University Press, 1989.

[Wit] Edward Witten. Monopoles and 4-manifolds. *Math. Res. Letters*, 1:769–796, 1994.